MAU22200 Lecture 20

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Images of compact sets are compact

If X is compact and $f : X \to Y$ is continuous then $f_*(X)$ is compact.
Suppose $\mathcal{G}$ is an open cover of $f_*(X)$. The set $\mathcal{H}$ of U such that $U = f^*(V)$ for some $V \in \mathcal{G}$ is an open cover of X. This can be written more succinctly as $\mathcal{H} = (f^*)_* (\mathcal{G})$. Each $U \in \mathcal{H}$ is open because f is continuous. They cover X because $f(x) \in f_*(X)$ so $f(x) \in V$ for some $V \in \mathcal{G}$ and therefore $x \in f^*(V) = U$ for the corresponding U.
X is compact, so there's a finite subcover $\mathcal{E}$ of $\mathcal{H}$. For each $U \in \mathcal{E}$ choose a $V \in \mathcal{G}$ such that $U = f^*(V)$. There might be more than one, but *choose only one*. Let $\mathcal{F}$ be the set of V you've chosen. It's a finite subcover of $\mathcal{G}$. It covers $f_*(X)$ because if $y \in f_*(X)$ then $y = f(x)$ for some $x \in X$ and $x \in U$ for some $U \in \mathcal{E}$. Then $x \in f^*(V)$ for some $V \in \mathcal{F}$. So $f(x) \in V$, i.e. $y \in V$.

A separation property

If X is Hausdorff, $A \in \wp(X)$ is compact and $y \in X \setminus A$ then there are open sets V and W such that $A \subseteq V$, $y \in W$ and $V \cap W = \varnothing$.
Note that the special case $A = \{x\}$ is just the definition of a Hausdorff topology.
The proof has a pattern which is used often.
For each $x \in A$ we have open V_x and W_x such that $x \in V_x$, $y \in W_x$ and $V_x \cap W_x = \varnothing$. There's an application of the Axiom of Choice hidden in the notation. The sets V_x for $x \in A$ form an open cover of A. A is compact so there's a finite subcover. In other words, we only need finitely many x's. There's a finite subset $F \subseteq A$ such that $A \subseteq \bigcup_{x \in F} V_x$. Set $V = \bigcup_{x \in F} V_x$ and $W = \bigcap_{x \in F} W_x$. W is open because it's an intersection of finitely many open sets. $A \subseteq V$. $y \in W$. If $z \in V \cap W$ then $z \in V_x$ for some $x \in F$, but then $z \in W_x$ and $V_x \cap W_x = \varnothing$.

Consequences

If X is Hausdorff, $A \in \wp(X)$ is compact and $y \in X \setminus A$ then there are open sets V and W such that $A \subseteq V$, $y \in W$ and $V \cap W = \varnothing$.
For any $y \in X \setminus A$ there is an open W_y such that $y \in W$ and $W_y \subseteq X \setminus A$. The union of all such W_y for $y \in X \setminus A$ is $X \setminus A$, which is therefore open, so A is closed.
If X is compact Hausdorff and $A, B \in \wp(X)$ are such that $A \cap B = \varnothing$ then there are open V and W such that $A \subseteq V$, $B \subseteq W$ and $V \cap W = \varnothing$.
A is compact. For each $y \in B$ choose open V_y and W_y such that $A \subseteq V_y$, $y \in W_y$ and $V_y \cap W_y = \varnothing$¿ The W_y cover B, so there's a finite $F \subseteq B$ such that $B \subseteq \bigcup_{y \in F} W_y$. Set $V = \bigcap_{y \in F} V_y$ and $W = \bigcup_{y \in F} W_y$. V is open because it's an intersection of finitely many open sets. $A \subseteq V$. $B \subseteq W$. $V \cap W = \varnothing$.

Intersections

If $\mathcal{K}$ is a set of compact subsets of a Hausdorff space then $\bigcap_{K\in\mathcal{K}} K$ is compact.
Each $K \in \mathcal{K}$ is closed, by the previous slide, so their intersection is closed. $\bigcap_{K\in\mathcal{K}} K \subseteq L$ for any $L \in \mathcal{K}$. So $\bigcap_{K\in\mathcal{K}} K \subseteq L$ is a closed subset of a compact space, and hence compact.
$(X, \mathcal{T})$ is compact if and only if whenever $\mathcal{C}$ is a set of closed subsets of X such that $\bigcap_{V\in\mathcal{E}} V \neq \varnothing$ for all finite $\mathcal{E} \subseteq \mathcal{C}$ we have $\bigcap_{V\in\mathcal{C}} V \neq \varnothing$.
This is really just a restatement of the definition in using complements, but it's a useful restatement. It's most often applied to $\mathcal{C} = \{K_1, K_2, \ldots\}$ where the K's are a nested sequence of non-empty compact subsets, i.e. $K_1 \supseteq K_2 \supseteq \cdots$.

Tychonoff (1/2)

Tychonoff's Theorem says that *any* product of compact sets is compact. This is actually equivalent to Zorn's Lemma or the Axiom of Choice. I'm not aware of any short proof.

The special case of the product of two compact sets is much easier. You can get finite products from this case by induction on the number of factors.

Suppose then that X and Y are compact and let $P = X \times Y$. Suppose $\mathcal{G}$ is an open cover of P. If $(x, y) \in P$ then $(x, y) \in Z$ for some $Z \in \mathcal{G}$. There must be $V \in \mathcal{T}_X$ and $W \in \mathcal{T}_Y$ such that $x \in V$, $y \in W$ and $V \times W \subseteq Z$. Choose such Z, V and W for each $(x, y) \in P$ and call your choices $Z_{x,y}$, $V_{x,y}$ and $W_{x,y}$. For fixed x the sets $W_{x,y}$ as y ranges over Y cover Y. It has a finite subcover, so there's a finite $F_x \in \wp(X)$ such that $Y = \bigcup_{y \in F_x} W_{x,y}$. Let $U_x = \bigcap_{y \in F_x} V_{x,y}$. U_x is an open neighbourhood of x. The set of U_x as x ranges over X are an open cover of X, and so have a finite subcover. So there is a finite $E \in \wp(X)$ such that $X = \bigcup_{x \in E} U_x$.

Tychonoff (2/2)

I claim that the set of $Z_{x,y}$ where $x \in E$ and $y \in F_x$ are a finite subcover of $\mathcal{G}$. It's a finite union of finite sets, so it's certainly finite. Each $Z_{x,y} \in \mathcal{G}$, so it's a subset. What's harder to show is that it's a cover.
Suppose $(s,t) \in P$. $s \in X$ so $s \in U_x$ for some $x \in E$. Then $t \in W_{x,y}$ for some $y \in F_x$. $s \in V_{x,y}$ so $(s,t) \in V_{x,y} \times W_{x,y}$. $V_{x,y} \times W_{x,y} \subseteq Z_{x,y}$ so $(s,t) \in Z_{x,y}$. So every element of P belongs to some $Z_{x,y}$ where $x \in E$ and $y \in F_x$.

The Heine-Borel Theorem

The Heine-Borel Theorem says that subsets of $\mathbf{R}^n$ are compact if and only if they are closed and bounded. This gives us many examples of compact spaces. We saw a special case in Lecture 19: Intervals in $\mathbf{R}$ are compact if and only if they are empty or of the form $[a, b]$. We'll use that special case together with most of the properties from the these two lectures to prove Heine-Borel.
$\mathbf{R}^n$ is a metric space, hence Hausdorff, and so all compact subsets are closed. If $A \in \mathbf{R}^n$ is not bounded then the sets $B(0, r)$ for $r > 0$ form an open cover of A with no finite subcover, so A must be bounded. This establishes the "only if" part.
Suppose A is closed and bounded. Bounded means it's contained in $B(0, r)$ for some $r > 0$. Therefore it's contained in the product of n copies of the interval $[-r, r]$. These intervals are compact and so is their product. A is a closed subset of a compact space and so is compact. This establishes the "if" part.