MAU22200 Lecture 19

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Compactness (definitions)

The concept of compactness is very important, but the definitions are opaque. You hope not to use them most of the time, but use their various properties instead.

- An open cover of $(X, \mathcal{T}_X)$ is a $\mathcal{G} \subseteq \mathcal{T}_X$ such that $X = \bigcup_{U \in \mathcal{G}} U$.
- $\mathcal{F}$ is a *subcover* of $\mathcal{G}$ if $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{F}$ is an open cover.
- $(X, \mathcal{T}_X)$ is *compact* if every open cover of X has a finite subcover.
- A subset A of X is compact if $(A, \mathcal{T}_A)$ is compact, where $\mathcal{T}_A$ is the subspace topology on A.
- A is *relatively compact* if its closure is compact.
- X is *σ-compact* if it is the union of countably many compact subsets.

Subsets

There's an alternate notion of open covers for subsets.

- $\mathcal{G}$ is an open cover of A with respect to $(X, \mathcal{T}_X)$ if $\mathcal{G} \subseteq \mathcal{T}_X$ and $A \subseteq \bigcup_{U \in \mathcal{G}} U$.

This is not the same as an open cover of $(A, \mathcal{T}_A)$, which is a subset of $\mathcal{T}_A$.

A is compact if and only if every open cover of A with respect to $(X, \mathcal{T}_X)$ has a finite subcover. This is a proposition, not a definition, since we already defined what it meant for A to be compact. Once the proposition is proved you can use it as if it were a definition though.

Note that compactness is still an absolute property though, not a relative one. Whether a subset is compact depends only on its topology, not on what space it's a subset of.

Examples

- Every finite space is compact, because open covers can only be finite.
- Discrete spaces are compact if and only if they are finite. If X is infinite the set of all sets $\{x\}$ for $x \in X$ is an open cover with no finite subcover.
- Intervals in $\mathbf{R}$ are compact if and only if they're empty or of the form $[a, b]$.
  - The empty interval is finite, so is compact.
  - $\mathbf{R}$ and the semi-infinite intervals $[a, +\infty)$, $(a, +\infty)$, $(-\infty, b]$ and $(-\infty, b)$ are not compact because the set of all sets $(-r, r)$ for $r > 0$ is an open cover without a finite subcover.
  - The intervals (a, b), $[a, b)$ and $(a, b]$ are not compact because the sets $(-\infty, y)$ with $y < b$ or $(x, +\infty)$ with $x > a$ form an open cover without a finite subcover.
  - The only thing left to prove is that $[a, b]$ is compact.

$[a, b]$ is compact (1/2)

Assume $[a, b]$ is not compact, i.e. that it has an open cover $\mathcal{G}$ which has no finite subcover. $\mathcal{G}$ is an open cover of $\left[a, \frac{a+b}{2}\right]$ and of $\left[\frac{a+b}{2}, b\right]$. If $\mathcal{F}_l$ was a finite subcover for $\left[a, \frac{a+b}{2}\right]$ and $\mathcal{F}_r$ was a finite subcover for $\left[\frac{a+b}{2}, b\right]$ then $\mathcal{F}_l \cup \mathcal{F}_r$ would be a finite subcover for $[a, b]$, but we assumed there wasn't one, so at least one of the intervals has no finite subcover of $\mathcal{G}$. Suppose it's $\left[a, \frac{a+b}{2}\right]$. $\mathcal{G}$ is an open cover of $\left[a, \frac{3a+b}{4}\right]$ and of $\left[\frac{3a+b}{4}, \frac{a+b}{2}\right]$. It has no finite subcover for at least one of these. Suppose it's $\left[\frac{3a+b}{4}, \frac{a+b}{2}\right]$. $\mathcal{G}$ is an open cover of $\left[\frac{3a+b}{4}, \frac{5a+3b}{8}\right]$ and of $\left[\frac{5a+3b}{8}, \frac{a+b}{2}\right]$. ...

$[a, b]$ is compact (2/2)

Continuing in this way we get a sequence of subintervals of $[a, b]$, for none of which does $\mathcal{G}$ have a finite subcover. The k'th interval is of length $\frac{b-a}{2^k}$. The left endpoints form an increasing sequence while the right endpoints form an decreasing sequence. They have a common limit, which is in $[a, b]$. Let $x \in [a, b]$ be this common limit. $x \in [a, b]$ so $x \in U$ for some $U \in \mathcal{G}$. U is open so there's an $\delta > 0$ such that $B(x, \delta) \subseteq U$. For k sufficiently large $\frac{b-a}{2^k} < \delta$, so the k'th interval is contained in U. But then $\{U\}$ is a finite cover, so we have a contradiction. Therefore $[a, b]$ is compact.

Properties

Usually we try to avoid using the definition directly. Instead we use various properties of compact sets.

- Closed subsets of compact sets are compact.
- Finite unions of compact subsets are compact.
- The image of a compact set under a continuous function is compact.
- Compact subsets of a Hausdorff space are closed.
- Products of compact spaces are compact.
- Subsets of $\mathbf{R}^n$ are compact if and only if they are closed and bounded.
- Continuous real valued functions on a compact set have a minimum and maximum.

All of these are proved in the notes, except the statement for products is proved only for finite products. I will also cover most of the proofs in lecture.

Two proofs

Closed subsets of compact sets are compact: Suppose $(X, \mathcal{T}_X)$ is compact and $A \in \wp(X)$ is closed.
If $\mathcal{G}$ is an open cover of A then $\mathcal{G} \cup \{X \setminus A\}$ is an open cover of X. X is compact so $\mathcal{G} \cup \{X \setminus A\}$ has a finite subcover $\mathcal{F}$. $\mathcal{F}$ is a finite open cover of A, but possibly not a subset of $\mathcal{G}$, so may not be a finite subcover, but $\mathcal{F} \setminus \{X \setminus A\}$ is a finite subcover.
Finite unions of compact subsets are compact: If $(X, \mathcal{T}_X)$ is a topological space and $K_1, \ldots, K_m \in \wp(X)$ are compact then $\bigcup_{j=1}^{m} K_j$ is compact.
If $\mathcal{G}$ is an open cover of $\bigcup_{j=1}^{m} K_j$ then it's also an open cover of K_j for each j. K_j is compact so there's a finite $\mathcal{F}_j \subseteq \mathcal{G}$ such that $\mathcal{F}_j$ is an open cover of K_j. Then $\bigcup_{j=1}^{m} \mathcal{F}_j$ is an open cover of $\bigcup_{j=1}^{m} K_j$. It's finite. It's a subset of $\mathcal{G}$. So it's a finite subcover of $\bigcup_{j=1}^{m} K_j$.