MAU22200 Lecture 18

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Connectedness and path connectedness

A topological space $(X, \mathcal{T})$ is called *path connected* if for every $x, y \in X$ there is a continuous $p \colon [0, 1] \to X$ such that $p(0) = x$ and $p(1) = y$.

This condition is fairly easy to understand, and often easy to check, but ultimately not as useful the following:

A topological space $(X, \mathcal{T})$ is called *connected* if there are no non-empty $V, W \in \mathcal{T}$ such that $V \cap W = \varnothing$ and $V \cup W = X$.

Every path connected space is connected, but not every connected space is path connected.

Both of these concepts can be applied to subsets, by equipping them with the subspace topology.

The definition of connected is about the non-existence of two sets, which is hard to check directly. Statements about connectedness are often proved by contradiction, in some form or other.

Intervals are connected

Intervals in **R** are connected. Conversely, any connected subset of **R** is an interval.

The proof that intervals are connected is by contradiction, but still somewhat constructive. Suppose there are non-empty $V, W \in \mathcal{T}$ such that $V \cap W = \varnothing$ and $V \cup W = X$. Choose $x \in V$ and $y \in W$. Suppose, without loss of generality, that $x < y$. Start from the interval $[x, y]$, which has one endpoint in V and one in W. Its midpoint is either in V or W. If we split the interval in two then one of the halves has one endpoint in V and one in W. We can repeat this infinitely many times. The endpoints converge to a common limit. V and W are both open, but also both closed, since each is the relative complement of the other. So the limit is in V and W. This is our contradiction.

The proof that path connected sets are connected is also by contradiction, showing that any counter-example would imply that $[0, 1]$ is disconnected.

Properties of connected sets

In the notes I'm careful to indicate spaces as sets *plus* topologies on them, but increasingly I'll omit the topology on the slides, except where we have more than one topology on the same space.

1. If $f : X \to Y$ is a continuous surjection and X is connected then Y is connected.
2. If $X = A \cup B$ where A and B are connected and $A \cap B \neq \varnothing$ then X is connected.
3. The product of an indexed collection of sets is connected if and only if each of the sets in the collection is connected.
4. If A is a connected subset of a topological space then the closure of A is also connected.

1 and 2 are fairly easy. 4 is a bit harder. The "only if" part of 3 is an easy consequence of 1, but the "if" part is hard.

Proof of 1

If $f : X \to Y$ is a continuous surjection and X is connected then Y is connected.
As mentioned, disconnectedness is simpler than connectedness, so we prove that if Y is disconnected then X is disconnected.
So we suppose Y is disconnected. In other words, that V and W are non-empty open sets such that $V \cap W = \varnothing$ and $V \cup W = Y$.
Then $f^*(V)$ and $f^*(W)$ are non-empty open sets. Non-empty because f is a surjection, and open because f is continuous.

$$f^*(V) \cap f^*(W) = f^*(V \cap W) = f^*(\varnothing) = \varnothing.$$

$$f^*(V) \cup f^*(W) = f^*(V \cup W) = f^*(Y) = X.$$

So X is disconnected.

Proof of 2

If $X = A \cup B$ where A and B are connected and $A \cap B \neq \varnothing$ then X is connected.
Again, the proof is indirect. One shows that if $X = V \cup W$ and $V \cap W = \varnothing$ for open V and W then at most one of V or W is non-empty.
Choose $x \in A \cap B$. Suppose, without loss of generality, that $x \in V$. $A \cap V$ and $A \cap W$ are open in the subspace topology on A and $A \cap V$ is non-empty. $(A \cap V) \cap (A \cap W) = \varnothing$, $(A \cap V) \cup (A \cap W) = A$, and A is connected so $A \cap W$ is empty.
Similarly, $B \cap W$ is empty.

$$W = X \cap W = (A \cup B) \cap W = (A \cap W) \cup (B \cap W) = \varnothing.$$

At most one of V or W is non-empty, so X is connected.

Proof of 4

If A is a connected subset of a topological space then the closure of A is also connected.
We show instead that if $\overline{A}$ is disconnected then A is disconnected.
So we suppose $\overline{A} = (\overline{A} \cap V) \cup (\overline{A} \cap W)$ where V and W are open and $(\overline{A} \cap V) \cap (\overline{A} \cap W) = \varnothing$.
$(A \cap V) \cap (A \cap W) \subseteq (\overline{A} \cap V) \cap (\overline{A} \cap W)$ and
$(\overline{A} \cap V) \cap (\overline{A} \cap W) = \varnothing$ so $(A \cap V) \cup (A \cap W) = \varnothing$.

$$\begin{aligned}
(A \cap U) \cup (A \cap V) &= A \cap (U \cup V) \\
&= (A \cap \overline{A}) \cap (U \cup V) \\
&= A \cap (\overline{A} \cap (U \cup V)) \\
&= A \cap \left((\overline{A} \cap U) \cup (\overline{A} \cap V)\right) \\
&= A \cap \overline{A} = A.
\end{aligned}$$

Proof of 4, continued

So $A \cap V$ and $A \cap W$ are open, $(A \cap V) \cap (A \cap W) = \varnothing$ and $(A \cap V) \cup (A \cap W) = A$. Are we done?
No, we still need to show that $A \cap V$ and $A \cap W$ are non-empty!
$\overline{A} \cap V$ and $\overline{A} \cap W$ are non-empty, but how is this helpful?
Choose $x \in \overline{A} \cap V$ and $y \in \overline{A} \cap W$. Remember the many properties of interiors, closures and boundaries. $x \in \overline{A}$ if and only if for all $V \in \mathcal{O}(x)$ the set $A \cap V$ is non-empty! Similarly $y \in \overline{A}$ so $A \cap W$ is non-empty, and we're done.

A connected but not path connected set

Consider the sets

$$S_\alpha = \{(x, y) \in \mathbf{R}^2 \colon \exists\theta \in \mathbf{R} \colon x = r(\alpha, \theta)\cos(\theta), y = r(\alpha, \theta)\sin(\theta)\},$$

where

$$r(\alpha, \theta) = \exp(\alpha \exp(-\theta)).$$

$$\lim_{\theta \to +\infty} r(\alpha, \theta) = 1.$$

$r(\alpha, \theta)$ is a monotone decreasing function of θ if $\alpha > 0$, monotone increasing if $\alpha < 0$ and constant if $\alpha = 0$. $S_{-1} \cup S_0 \cup S_1$ is connected, but not path connected. We can't separate S_0 from S_{-1} or S_1 by open sets because every point of S_0 is a limit point of S_{-1} and of S_1, but there's no path from S_{-1} to S_1.