MAU22200 Lecture 17

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The product topology

Recall the construction of products of sets. We start from an indexed collection of sets, i.e. a function $j\colon L \to \mathcal{A}$. The elements of the product are choice functions, i.e. functions $f\colon L \to \bigcup_{\lambda \in L} j(\lambda)$ such that $f(\lambda) \in j(\lambda)$. There are projections π_λ from the product to $j(\lambda)$, given by $\pi_\lambda(f) = f(\lambda)$.

Suppose each $j(\lambda)$ has a topology $\mathcal{T}_\lambda$. What topology should the product have? You could guess, but you'd almost certainly guess "wrong".

How can a definition be wrong? If the topology you've guessed either isn't a topology or doesn't have many useful properties. The most obvious guess fails to be a topology and the second most obvious guess has few interesting properties.

Instead we define the subspace topology to be weakest topology among those which make all the projections are continuous.

Topologies from sets of functions

Last time we had a proposition which told us how to get topologies from functions.

There's an analogue, or rather two analogues, for sets of functions.

Suppose X is a set and $j\colon L \to \mathcal{Y}$ is an indexed collection of sets. Suppose for each $\lambda \in L$ we have a topology $\mathcal{T}_\lambda$ on $j(\lambda)$ and a function $f_\lambda\colon X \to j(\lambda)$. Then there is a weakest topology $\mathcal{T}_X$ on X such that f_λ is continuous with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_\lambda$ on X and $j(\lambda)$ for each $\lambda \in L$. This topology is generated by $\bigcup_{\lambda \in L} \left(f_\lambda^*\right)_* (\mathcal{T}_\lambda)$.

Suppose Y is a set and $i\colon K \to \mathcal{X}$ is an indexed collection of sets. Suppose for each $\kappa \in K$ we have a topology $\mathcal{T}_\kappa$ on on $i(\kappa)$ and a function $f\colon i(\kappa) \to Y$. Then there is a strongest topology $\mathcal{T}_Y$ on Y which makes f_κ continuous for each $\kappa \in K$. This topology is $\bigcap_{\kappa \in K} f_\kappa^{**}(\mathcal{T}_\kappa)$.

A more concrete description

The first proposition from the last slide tells us that the product topology exists, and more or less what it is. Suppose F is a finite subset of L and t is a function from F to $\cup_{\lambda \in F} \mathcal{T}_\lambda$ such that $t(\lambda) \in \mathcal{T}_\lambda$ for each $\lambda \in F$. Let $U_{F,t}$ be the set of choice functions f such that $f(\lambda) \in t(\lambda)$ for all $\lambda \in F$. The product topology consists of all unions of sets of the form $U_{F,t}$. The restriction to *finite* subsets is redundant if L is finite but is important if L is infinite. Usually $\#L = 2$ and we're constructing a topology on the Cartesian product. The open sets on $X \times Y$ are unions of products of an open set in X with and open set in Y.

Projections are open

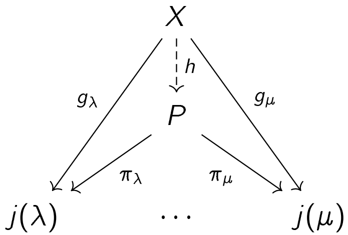
A function f from a topological space $(X, \mathcal{T}_X)$ to a topological space $(Y, \mathcal{T}_Y)$ is called *open* if $f_*(U) \in \mathcal{T}_Y$ whenever $U \in \mathcal{T}_X$. Note that this is not the same as continuity, which would be $f^*(V) \in \mathcal{T}_X$ whenever $V \in \mathcal{T}_Y$. The projections from a product are continuous because that's how we defined the product topology, but they are also open.

$$\pi_\lambda(U_{F,t}) = \begin{cases} t(\lambda) & \text{if } \lambda \in F \\ j(\lambda) & \text{if } \lambda \notin F. \end{cases}$$

In either case $\pi_\lambda(U_{F,t}) \in \mathcal{T}_\lambda$. The image of union is the union of the images and unions of elements of $\mathcal{T}_\lambda$ are elements of $\mathcal{T}_\lambda$, so the image of any element of $\mathcal{T}_P$ is an element of $\mathcal{T}_\lambda$. In other words, π_λ is open.

Another diagram (1/2)

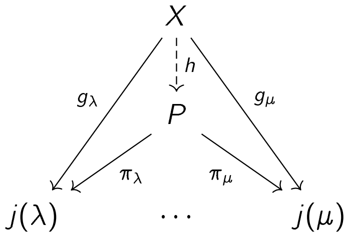
Given a topological space $(X, \mathcal{T}_X)$ and functions $g_\lambda \colon X \to j(\lambda)$ for each $\lambda \in L$, there is a unique function h from X to the product of j such that $\pi_\lambda \circ h = g_\lambda$. This h is continuous.



The existence and uniqueness of h are easy. Take $x \in X$ and let $f = h(x)$. $f(\lambda) = \pi_\lambda(f) = (\pi_\lambda \circ h)(x) = g_\lambda(x)$. This specifies $f(\lambda)$ uniquely for each $\lambda \in L$, and so specifies $h(x)$ uniquely for each $x \in X$. Conversely, if we define h by the formula above then $\pi_\lambda \circ h = g_\lambda$ for each λ.

Another diagram (2/2)

The hard part of the proposition is the continuity statement.



It would be easy to get the continuity of g_λ from that of π_λ and h but we want to get the continuity of h from that of π_λ and g_λ. This is where we need the fact that the product topology is the weakest topology which makes the projections continuous. See the notes for details.

The diagonal function and set

In the case where $j(\lambda) = X$ and $g_\lambda(x) = x$ for all $\lambda \in L$ the function h then is the *diagonal function*, sometimes called the diagonal embedding. Its image is called the diagonal subset. In the case of $X \times X$ the diagonal function is $h(x) = (x, x)$ and the diagonal set is $\Delta_X = \{(x, y) \in X \colon x = y\}$.
The diagonal function is continuous as a consequence of the theorem If we had defined the product topology in terms of unions sets of the form $U_{F,t}$ where the sets F were allowed to be infinite then the diagonal function would not be continuous in general when L is infinite, even if all the factors in the product are $\mathbf{R}$.

Products and Hausdorff spaces

The diagonal can be used to give a simple criterion from Hausdorffness: X is Hausdorff if and only if Δ_X is a closed subset of $X \times X$.

The product of an indexed collection of Hausdorff spaces is Hausdorff.

Conversely, if the product of an indexed collection of non-empty spaces is Hausdorff then each of the spaces is Hausdorff.

None of these statements is obvious, or particularly easy to prove.

Disjoint unions

Recall the following from an earlier slide:

> *Suppose Y is a set and $i\colon K \to \mathcal{X}$ is an indexed collection of sets. Suppose for each $\kappa \in K$ we have a topology $\mathcal{T}_\kappa$ on on $i(\kappa)$ and a function $f\colon i(\kappa) \to Y$. Then there is a strongest topology $\mathcal{T}_Y$ on Y which makes f_κ continuous for each $\kappa \in K$.*

This can be used to put a topology on the disjoint union of an indexed collection of sets. We choose the strongest topology such that each inclusion function is continuous.
We'll skip over the properties of the disjoint union topology.