MAU22200 Lecture 16

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Topologies from functions

Suppose $f : X \to Y$

- there is, for any topology $\mathcal{T}_Y$ on Y, a weakest topology $\mathcal{T}_X$ on X among those which make f continuous, and
- there is, for any topology $\mathcal{T}_X$ on X, a strongest topology $\mathcal{T}_Y$ on Y among those which make f continuous.

There's also a strongest $\mathcal{T}_X$ and weakest $\mathcal{T}_Y$, but that's less interesting: they're the discrete topology on X and the trivial topology on Y.

These topologies can be described more explicitly. In the first part $\mathcal{T}_X = (f^*)_*(\mathcal{T}_Y)$ while in the second part $\mathcal{T}_Y = f^{**}(\mathcal{T}_X)$.

This is often used to construct topologies, since it's often clearer which functions should be continuous than which sets should be open.

Subspace topology

Suppose that $(X, \mathcal{T}_X)$ is a topological space and $A \subseteq X$. What topology should A have?
You could guess, and you'd probably guess correctly.
Or you could use the proposition and define $\mathcal{T}_A$ to be the weakest topology on A for which the inclusion $i \colon A \to X$, defined by $i(x) = x$, is continuous. We call this the *subspace topology* on A.
You still want a more explicit description though. The proposition which tells us there's a weakest topology which makes i continuous also tells us what it is.

$$\mathcal{T}_A = (i^*)_*(\mathcal{T}_X).$$

$U \in (i^*)_*(\mathcal{T}_X)$ if and only if $U = (i^*)(V)$ for some $V \in \mathcal{T}_X$.
$x \in (i^*)(V)$ means $i(x) \in V$ for some $x \in A$. $i(x) = x$. So $x \in (i^*)(V)$ iff $x \in A$ and $x \in V$, i.e. $x \in A \cap V$. In other words, $i^*(V) = A \cap V$. So $\mathcal{T}_A$ is set of sets of the form $A \cap V$ where $V \in \mathcal{T}_X$.

Open and closed are relative terms

Note that open and closed are relative terms, not absolute! It doesn't make sense to say that U is open or closed, only that it is an open or closed subset of some larger set, equipped with some topology. We can only omit the reference to the containing set or its topology if they are obvious from context.

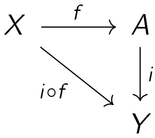
For example, $(0, 1]$ is not an open subset of $\mathbf{R}$, with the usual topology, but it is an open subset of $[-1, 1]$ with the subspace topology, since

$$(0, 1] = [-1, 1] \cap (0, 2)$$

and $(0, 2)$ is an open subset of $\mathbf{R}$. Similarly $[0, 1)$ is a closed subset of $(-1, 1)$, with the subspace topology, but not of $\mathbf{R}$, with the usual topology.

Properties of the subspace topology (1/2)

Suppose $A \subseteq Y$ and $f : X \to A$ is a function. Then f is continuous if and only if $i \circ f$ is continuous. It's helpful when reading statements about compositions to draw diagrams like this.



For historical reasons they're called commutative diagrams.
"only if": The topology $\mathcal{T}_A$ was defined to make i continuous and compositions of continuous functions are continuous.
"if": If $W \in \mathcal{T}_Y$ then $(i \circ f)^*(W) \in \mathcal{T}_X$.
$(i \circ f)^*(W) = f^*(i^*(W))$. $i^*(W) = A \cap W$. So if $W \in \mathcal{T}_Y$ then $f^*(A \cap W) \in \mathcal{T}_X$. Every element of $\mathcal{T}_A$ is $A \cap W$ for some $W \in \mathcal{T}_Y$. So f is continuous.

Properties of the subspace topology (2/2)

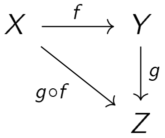
Any subspace of a Hausdorff space is Hausdorff.
Remember that $(S, \mathcal{T})$ is Hausdorff if for any $x \neq y$ there are $U \in \mathcal{T}$, $V \in \mathcal{T}$ such that $x \in U$, $y \in V$, $U \cap V = \varnothing$.
Suppose $(X, \mathcal{T}_X)$ is Hausdorff $A \subseteq X$ and $\mathcal{T}_A$ is the subspace topology on A. For any $x \neq y$ there are $U \in \mathcal{T}_X$, $V \in \mathcal{T}_X$ such that $x \in U$, $y \in V$, $U \cap V = \varnothing$. If $x, y \in A$ then $x \in A \cap U$, $y \in A \cap V$. $A \cap U \in \mathcal{T}_A$ and $A \cap V \in \mathcal{T}_A$. Also $(A \cap U) \cap (A \cap V) = A \cap (U \cap V) = \varnothing$. So $(A, \mathcal{T}_A)$ is Hausdorff.
We could also say that Hausdorff iff for any $x \neq y$ there are $U \in \mathcal{O}(x)$, $V \in \mathcal{O}(y)$ such that $U \cap V = \varnothing$. The notation for sets of neighbourhoods is convenient, but doesn't mention the topological space, so it would have been ambiguous here.

Quotient topology

Suppose that $(X, \mathcal{T}_X)$ is a topological space and $\sim$ is an equivalence relation on X. Let $\mathcal{E}$ be the set of equivalence classes with respect to $\sim$. What topology should $\mathcal{E}$ have?
You could guess, and you might guess correctly, or you could use the proposition from the first slide.
There's a natural function f from X to $\mathcal{E}$ which takes each $x \in X$ to its equivalence class. We choose the topology on $\mathcal{E}$ to be the strongest topology on $\mathcal{E}$ such that f is continuous. This is called the *quotient topology* on $\mathcal{E}$.
One can define it in more generality, for any function $f \colon X \to Y$. Y needn't be the set of equivalence classes of an equivalence relation $\sim$ and f needn't be the function taking elements to their equivalence classes. The extra generality is mostly an illusion though.

Properties of the quotient topology

Given $f\colon X \to Y$ and $\mathcal{T}_X$ the quotient topology $\mathcal{T}_Y$ is the strongest topology such that f is continuous. $U \in \mathcal{T}_Y$ if and only if $f^*(U) \in \mathcal{T}_X$.
If $f\colon X \to Y$ and Y is given the quotient topology then $g\colon Y \to Z$ is continuous if and only if $g \circ f$ is continuous.



"only if": f is continuous by construction so if g is continuous then $g \circ f$ is continuous.
"if": The quotient topology is $\mathcal{T}_Y = f^{**}(\mathcal{T}_X)$.
$f^*(V) \in \mathcal{T}_X \Leftrightarrow V \in f^{**}(\mathcal{T}_X) \Leftrightarrow V \in \mathcal{T}_Y$. If $V = g^*(W)$ and $W \in \mathcal{T}_Z$ then $f^*(V) = (g \circ f)^*(W) \in \mathcal{T}_X$ since $g \circ f$ is continuous. So $V \in \mathcal{T}_Y$. If $V = g^*(W)$ and $W \in \mathcal{T}_Z$ then $V \in \mathcal{T}_Y$. In other words, g is continuous.

Non-properties of the quotient space

Unlike the subspace case, we don't generally get a Hausdorff topology on the quotient space, even if we started from a Hausdorff topology on the domain of f. There's an example in the notes with $f : \mathbf{R} \to \mathbf{R}/\mathbf{Q}$. The quotient topology on $\mathbf{R}/\mathbf{Q}$ is the trivial topology, which is not Hausdorff.
There aren't many useful theorems which don't have Hausdorff in their hypotheses, so the quotient topology is less useful than the subspace topology.