MAU22200 Lecture 15

John Stalker

Trinity College Dublin

14 October 2021

Stronger and weaker

So far we've considered a single topology on a single set. Now we'll consider multiple topologies a single set and functions between sets, each with its own topology.
I'll write $\mathbf{T}(X)$ for the set of topologies on X.
For example, if $X = \{1, 2\}$ then $\mathbf{T} = \{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ where $\mathcal{T}_0 = \{\varnothing, \{1, 2\}\}$, $\mathcal{T}_1 = \{\varnothing, \{1\}, \{1, 2\}\}$, $\mathcal{T}_2 = \{\varnothing, \{2\}, \{1, 2\}\}$, and $\mathcal{T}_3 = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. $\mathcal{T}_0$ is the trivial topology and $\mathcal{T}_3$ is the discrete topology. There are various inclusion relations:
$\mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \mathcal{T}_3$ and $\mathcal{T}_0 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$.
In general if $\mathcal{W}, \mathcal{S} \in \mathbf{T}(X)$ and $\mathcal{W} \subseteq \mathcal{S}$ then we say that $\mathcal{W}$ is weaker than $\mathcal{S}$ and $\mathcal{S}$ is stronger than $\mathcal{W}$.
In the example, $\mathcal{T}_1$ is stronger than $\mathcal{T}_0$ and weaker than $\mathcal{T}_3$. $\mathcal{T}_1$ is neither stronger nor weaker than $\mathcal{T}_2$.

Examples

In general, every topology is stronger than the trivial topology and weaker than the discrete topology. Also, every topology is stronger and weaker than itself! As with relations, the terminology suggests a strict comparison, but the definition has a non-strict comparison. If we want a strict comparison we have to say "strictly stronger" or "strictly weaker".

The usual (metric) topology on $\mathbf{R}^n$ or $\mathbf{C}^n$ is strictly stronger than the Zariski topology. Function spaces often have multiple interesting topologies, with some weaker than others. Consider the space $C([a,b])$ of continuous functions on $[a,b]$ with $a < b$. For $p \in [1,+\infty)$ the norm

$$\|f\|_p = \left(\int_{[a,b]} |f(x)|^p \, dx\right)^{1/p}$$

gives a topology $\mathcal{T}_p$ on $C([a,b])$. If $p < q$ then $\mathcal{T}_p$ is strictly weaker than $\mathcal{T}_q$.

Intersections and unions

The intersection of any set of topologies on X is a topology on X. It's stronger than any of them. The union of topologies is generally not a topology.
$\mathcal{T}_1 = \{\varnothing, \{1\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\}$ and
$\mathcal{T}_2 = \{\varnothing, \{2\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}$ are topologies on $\{1,2,3,4\}$. $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology, because it contains $\{1,3\}$ and $\{2,3\}$ but not their intersection $\{3\}$. It also contains $\{1\}$ and $\{2\}$ but not their union $\{1,2\}$. There is a topology
$\mathcal{T} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}$
which contains, i.e. is stronger than, both $\mathcal{T}_1$ and $\mathcal{T}_2$. $\mathcal{T}$ is not the only topology stronger than $\mathcal{T}_1$ and $\mathcal{T}_2$. The discrete topology is also stronger than both, as are a number of other topologies. $\mathcal{T}$ is special in that it is weaker than all topologies stronger than $\mathcal{T}_1$ and $\mathcal{T}_2$. It was obtained by adding only $\{3\}$ and $\{1,2\}$, both of which had to be added.

Topology generated by a set of sets

Given a set $\mathbf{A}$ of topologies on X there is strongest topology among all those which are weaker than every element of $\mathbf{A}$. It's obtained by taking their intersection.

Given a set $\mathcal{A}$ of subsets of $\mathcal{A}$ there is a weakest topology among all topologies of which $\mathcal{A}$ is a subset. This is called the topology *generated by* $\mathcal{A}$. It's obtained by letting $\mathbf{A}$ be the set of topologies of which $\mathcal{A}$ is a subset and applying the result above.

In the example we had $\mathcal{A} = \mathcal{T}_1 \cup \mathcal{T}_2$, but $\mathcal{A}$ doesn't need to be a union of topologies. When $\mathcal{A}$ is a union of topologies we can describe the topology it generates as the set of unions of intersections of finitely many elements of $\mathcal{A}$.

Many topologies are, or can be, described by a set of generators. The open balls generate the usual topology on $\mathbf{R}^n$, or any metric space. The open balls with rational centre and radius generate the usual topology on $\mathbf{R}^n$.

Continuity

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces then $f : X \to Y$ is called *continuous* if for all $f^*(W) \in \mathcal{T}_X$ for all $W \in \mathcal{T}_Y$, or, equivalently, $\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}_X)$.

There's also a notion of continuity at a point. f is *continuous at x* if $f^*(Z) \in \mathcal{N}(x)$ for all $Z \in \mathcal{N}(f(x))$, or, equivalently, $\mathcal{N}(f(x)) \subseteq f^{**}(\mathcal{N}(x))$.

f is continuous if and only if it is continuous at x for all $x \in X$.

The composition of continuous functions is a continuous function. In fact if f is continuous at x and g is continuous at $f(x)$ then $g \circ f$ is continuous at x.

Dependence of continuity on choice of topology

Whether a function is continuous depends on the choice of topologies, but

> *Suppose $\mathcal{W}_X$ and $\mathcal{S}_X$ are topologies on X and $\mathcal{W}_X$ is weaker than $\mathcal{S}_X$. Suppose $\mathcal{W}_Y$ and $\mathcal{S}_Y$ are topologies on Y and $\mathcal{W}_Y$ is weaker than $\mathcal{S}_Y$. Suppose that $f : X \to Y$ is continuous with respect to the topologies $\mathcal{W}_X$ and $\mathcal{S}_Y$. Then it's also continuous with respect to the topologies $\mathcal{S}_X$ and $\mathcal{W}_Y$.*

Note that this is "weaker", not "strictly weaker". Usually one takes $\mathcal{W}_X = \mathcal{S}_X$ or $\mathcal{W}_Y = \mathcal{S}_Y$.