MAU22200 Lecture 14

John Stalker

Trinity College Dublin

11 October 2021

Rough plan of Chapter 3

- Review of definitions of topology, Hausdorff
- Properties of subsets, constructions on subsets
- Multiple topologies on a single set, stronger vs weaker
- Functions between topological spaces, topologies induced by functions
- Constructions: subspaces, products, quotients, disjoint unions
- Properties of spaces: connectedness, compactness, normality
- Useful theorems: Urysohn, Tietze, existence of partitions of unity

Interior and closure (properties)

The *interior* of a set is its largest open subset and the *closure* is its smallest closed superset. Their difference is the boundary. The interior, closure and boundary of A are written A°, $\overline{A}$ and ∂A. The definitions in the notes are slightly different, but equivalent. Example: $A = [0, 1) \subseteq \mathbf{R}$. Note that I had to specify what A was a subset of. These are *relative* notions. $A^\circ = (0, 1)$, $\overline{A} = [0, 1]$ and $\partial A = \overline{A} \setminus A^\circ = \{0, 1\}$.

1. A° is open, while $\overline{A}$ and ∂A are closed.
2. $A^\circ \subseteq A \subseteq \overline{A} \subseteq X$
3. If $A \subseteq B$ then $A^\circ \subseteq B^\circ$ and $\overline{A} \subseteq \overline{B}$.
4. If $A \subseteq B$ and A is open then $A \subseteq B^\circ$.
5. If $A \subseteq B$ and B is closed then $\overline{A} \subseteq B$.
6. A is open if and only if $A = A^\circ$.
7. B is closed if and only if $B = \overline{B}$.

Interior and closure (more properties)

8. $(A^\circ)^\circ = A^\circ$ and $\overline{(\overline{A})} = \overline{A}$.
9. $(X \setminus A)^\circ = X \setminus \overline{A}$ and $\overline{X \setminus A} = X \setminus A^\circ$.
10. $\partial (X \setminus A) = \partial A$.
11. The following three statements are equivalent:
    - 11.1 $x \in A^\circ$.
    - 11.2 There is a $W \in \mathcal{O}(x)$ such that $W \subseteq A$.
    - 11.3 There is a $W \in \mathcal{N}(x)$ such that $W \subseteq A$.
12. The following three statements are equivalent:
    - 12.1 $x \in \overline{A}$.
    - 12.2 For every $W \in \mathcal{O}(x)$, $W \cap A \neq \varnothing$.
    - 12.3 For every $W \in \mathcal{N}(x)$, $W \cap A \neq \varnothing$.
13. The following three statements are equivalent:
    - 13.1 $x \in \partial A$.
    - 13.2 For every $W \in \mathcal{O}(x)$, $W \cap A \neq \varnothing$ and $W \cap (X \setminus A) \neq \varnothing$.
    - 13.3 For every $W \in \mathcal{N}(x)$, $W \cap A \neq \varnothing$ and $W \cap (X \setminus A) \neq \varnothing$.

Balls

I used the notation $\bar{B}(x, r)$ for the closed ball of radius r about x.

$$\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}.$$

The open ball is

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

and its closure is $\overline{B(x, r)}$. Is $\bar{B}(x, r) = \overline{B(x, r)}$? Yes, in $\mathbf{R}^n$. No, in general. In particular, no in the discrete metric if $r = 1$. We do have $\overline{B(x, r)} \subseteq \bar{B}(x, r)$ in general though. Also, $B(x, r) \subseteq \bar{B}(x, r)^\circ$.

Closure and sequences/nets

We will see later that for metrisable spaces the following are equivalent:

- $z \in \overline{A}$.
- z is the limit of a sequence in A.

This isn't true in general, but it is true if we replace sequences by nets.
Recall that nets are functions from a directed set. If $f : D \to Y$ is a net then $\lim f = z$ means that for all $Z \in \mathcal{O}(z)$ there's an a in D such that $a \preccurlyeq b \Rightarrow f(b) \in Z$.
If $(D, \preccurlyeq) = (\mathbf{N}, \leq)$ then the net is a sequence. So if z is the limit of a sequence in A then $z \in \overline{A}$, even if the topology is not metrisable.

Dense subsets

$A \subseteq X$ is called *dense* if $\overline{A} = X$. For example, $\mathbf{Q}$ is a dense subset of $\mathbf{R}$. Every real number is the limit of a sequence of rational numbers, so $\overline{\mathbf{Q}} = \mathbf{R}$. The fact that $\mathbf{R}$ has a countable dense subset will be important later.

$\mathbf{Z}$ is not a dense subset of $\mathbf{R}$. $\mathbf{Z}$ is a closed subset and for any X the only closed dense subset of X is X itself. The following properties are proved in the notes:

1. If $A \subseteq B$ and A is dense then so is B.
2. The only dense closed set is X.
3. A is dense if any only if the interior of $X \setminus A$ is empty.
4. The following three statements are equivalent:
   - 4.1 A is dense.
   - 4.2 For every $x \in X$ and $W \in \mathcal{O}(x)$, $W \cap A \neq \varnothing$.
   - 4.3 For every $x \in X$ and $W \in \mathcal{N}(x)$, $W \cap A \neq \varnothing$.