# MAU22200 Lecture 13

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# Indexed collections of sets (1/3)

There are various types of products of sets we might consider, e.g.

- $X \times Y$
- $\mathbf{R}^n = \mathbf{R} \times \cdots \times \mathbf{R}$
- $\prod_{\mathbf{x} \in \mathbf{R}^n, r>0} B(\mathbf{x}, r)$.

The first example is straightforward, at least if $X \neq Y$. We can call the projection onto the first factor $\pi_X$ and the projection onto the second factor $\pi_Y$ if $X \neq Y$. $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$. This doesn't work if $X = Y$, by we can still call them $\pi_1$ and $\pi_2$. For $\mathbf{R}^n$ we don't want to call all $n$ projections $\pi_{\mathbf{R}}$. We can safely call them $\pi_1, \ldots, \pi_n$.

In the last example we can't number the factors since there are uncountably many. If we need to identify the projection onto a particular factor we could call it $\pi_{B(\mathbf{x},r)}$, or just $\pi_{\mathbf{x},r}$.

# Indexed collections of sets (2/3)

We need a way to identify the factor sets in a product which

- isn't just the sets, since there might be repeated factors, and
- isn't numbering them in order, since there could be uncountably many.

We didn't run into this problem for unions and intersections, even uncountable ones, because repeated sets didn't matter.

What we need is a set of labels. If $X \neq Y$ would could use $X$ and $Y$ as labels for the factors in $X \times Y$. If $X = Y$ we could label them as 1 and 2, or as $a$ and $b$ for any $a \neq b$. For $\mathbf{R}^n = \mathbf{R} \times \cdots \times \mathbf{R}$ the obvious labels are $1, \ldots, n$. For $\prod_{\mathbf{x} \in \mathbf{R}^n, r > 0} B(\mathbf{x}, r)$ we can label the factors by $\mathbf{x}$ and $r$. Here the label set is uncountable.

Allowing the set of labels to be arbitrary adds a layer of notation, but means we never need to worry about repeated factors or uncountably many factors.

# Indexed collections of sets (3/3)

An indexed collection of sets is a function $j$ from a label set $L$ to a set of sets $\mathcal{A}$. It serves to identify factor sets.
For $X \times Y$ with $X \neq Y$ you could take $L = \{X, Y\}$ and $j(\lambda) = \lambda$. Or you could take $L = \{1, 2\}$, $j(1) = X$ and $j(2) = Y$. This works even if $X = Y$.
For $\mathbf{R}^n$ the natural choice is $L = \{1, \ldots, n\}$ and $j(k) = \mathbf{R}$ for all $k \in L$.
For the product of balls we can take $L = \mathbf{R}^n \times \mathbf{R}_+$ and $j((\mathbf{x}, r)) = B(\mathbf{x}, r)$.
We first met the indexing problem for the disjoint union of two sets, but it applies to arbitrary disjoint unions and to arbitrary products. Products are more important, but disjoint unions are simpler.

# Disjoint unions

For a pair of sets $X$ and $Y$ with labels $a$ and $b$ the disjoint union of $X$ and $Y$ is the set of pairs $(a, x)$ for $x \in X$ or $(b, y)$ for $y \in Y$. For a general indexed collection of sets $j \colon L \to \mathcal{A}$ it's the pairs $(\lambda, x)$ where $\lambda \in L$ and $x \in j(\lambda)$.
For each label $\lambda \in L$ there's an inclusion function $i_\lambda$ from $j(\lambda)$ to the disjoint union defined by $i_\lambda(x) = (\lambda, x)$. The disjoint union is the union of the images all these inclusion functions. The images corresponding to distinct labels are disjoint, even if the corresponding sets intersect, or are the same. This is why it's called the disjoint union.

# Products

There's similar construction of products. You can think of $\mathbf{R}^n$ as a function from $\{1, \ldots, n\}$ to $\mathbf{R}$. If the factors are different it's more complicated. An element of $X \times Y$ "is" a function $f$ from $\{a, b\}$ such that $f(a) \in X$ and $f(b) \in Y$. What's it a function to? $X \cup Y$.

In general we want to define the product of an indexed collection of sets $j\colon L \to \mathcal{A}$ to be the set of functions from $f\colon L \to \bigcup_{X \in \mathcal{A}} X$ such that $f(\lambda) \in j(\lambda)$.

Just as we have inclusions in the disjoint union, we have projections from the product onto the factors. $\pi_\lambda(f) = f(\lambda)$.

Products are the main reasons we're considering indexed collections of sets. We'll want products of indexed collections of topological spaces. Of course we'll want a topology on that product. Roughly speaking, it will be chosen to make all the projections continuous.

# Choice

The functions which make up the product are called choice functions. If $j(\lambda) = \varnothing$ for some $\lambda \in L$ then there is no $f$ such that $f(\lambda) \in j(\lambda)$, so there are no choice functions and the product is empty.

What about the converse? Is the product of non-empty sets non-empty? This is, more or less, the Axiom of Choice: every indexed collection of non-empty sets has a choice function. For us it's not an axiom, but a theorem, proved using Zorn's Lemma. The proof in the notes is structured a bit differently from the other applications of Zorn's Lemma. We want a function from $L$ and the the proof starts from the empty function on the empty sets and gradually adds elements to the domain. It could have been structured like the other proofs, or they could have been structured like it. This version is probably more intuitive, but a bit less efficient.

# Topological spaces

Recall that a topological space is a pair $(X, \mathcal{T})$ where $\mathcal{T} \in \wp(\wp(X))$ is such that

- $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.
- If $V, W \in \mathcal{T}$ then $V \cap W \in \mathcal{T}$.
- If $\mathcal{E} \subseteq \mathcal{T}$ then $\bigcup_{V \in \mathcal{E}} V \in \mathcal{T}$.

We sometimes say $\mathcal{T}$ is "closed" under finite intersections and arbitrary unions. Unfortunately the word "closed" has an unrelated meaning in topology. The elements of $\mathcal{T}$ are called "open" sets and their (relative) complements are called "closed" sets.

For any metric space there is a natural topology, the one whose open sets are the ones which contain an open ball about each of their points. There are some interesting topologies not of this form, e.g. the Zariski topology. We can also get non-metrisable topologies from various constructions, e.g. products.

# Structure of notes

Chapter 3 is about general topological spaces. The first few sections are available online.
Eventually we mostly care about metric spaces, but we won't assume we have a metric until Chapter 4. Metric spaces are a good source of examples in Chapter 3 though. And we will treat one theorem, Heine-Borel, about normed vector spaces in this chapter.