MAU22200 Lecture 12

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Countable sets (1/2)

There are two different conventions for what countable means but the convention in this module is that X is countable if $\#X \leq \#\mathbf{N}$. It's called countably infinite if $\#X = \#\mathbf{N}$. Roughly, countable means not much larger than the natural numbers. Proper supersets of the natural numbers can be countable though. $\mathbf{Z}$ and $\mathbf{Q}$ are both countable.
$\mathbf{N} \times \mathbf{N}$ is also countable. The idea is simple enough. $\mathbf{N} \times \mathbf{N}$ is the set of points with integer coordinates in the first quadrant of the plane. They're all on a line $x + y = q$. Order them in increasing order of q, and in increasing order of y within each line, labeling each with the next available natural number. You'll label each point with a unique such number. You can write formulae for this. They're in the notes.

Countable sets (2/2)

There are various ways to prove a set is countable, if you can build it from other sets which are known to be countable.

1. *If $f : X \to Y$ is an injection and Y is countable then X is countable.*
2. *If $X \subseteq Y$ and Y is countable then X is countable.*
3. *If $g : Y \to X$ is a surjection and Y is countable then X is countable.*
4. *If X and Y are countable then so is $X \times Y$.*
5. *Suppose $\mathcal{A}$ is a countable set and then each $X \in \mathcal{A}$ is a countable set. Then $\cup_{X \in \mathcal{A}} X$ is countable.*
6. *If X and Y are countable then so is $X \cup Y$.*

You can use these, in combination, to show that the set of algebraic numbers is countable.

The Cantor Set (1/3)

The Cantor Set is the image of the function

$$f(A) = \frac{2}{3} \sum_{j \in A} 3^{-j}$$

from $\wp(\mathbf{N})$ to $\mathbf{R}$. $f(A) \geq 0$ because it's a sum of non-negative terms. Similarly, $f(\mathbf{N} \setminus A) \geq 0$.

$$f(A) + f(\mathbf{N} \setminus A) = \frac{2}{3} \sum_{j \in \mathrm{N}} 3^{-j} = 1.$$

So $0 \leq f(A) \leq 1$. Set $C = f_*(\wp(\mathbf{N}))$. Then $C \subseteq [0, 1]$.

The Cantor Set (2/3)

Define $s\colon \wp(\mathbf{N}) \to \wp(\mathbf{N})$ by

$$s(A) = \{i \in \mathbf{N} \colon i + 1 \in A\}.$$

E.g. $s(\{2, 3, 5, 7, \ldots\}) = \{1, 2, 4, 6, \ldots\}$.

$$f(s(A)) = \frac{2}{3} \sum_{i+1 \in A} 3^{-i} = \frac{2}{3} \sum_{\substack{j \in A \\ j > 0}} 3^{-j+1}.$$

$$f(A) = \begin{cases} \frac{1}{3} f(s(A)) & \text{if } 0 \notin A \\ \frac{2}{3} + \frac{1}{3} f(s(A)) & \text{if } 0 \in A \end{cases}$$

$x \in C$ if and only if $x = \frac{1}{3}y$ or $x = \frac{2}{3} + \frac{1}{3}y$ where $y \in C$. C is the union of two scaled copies of C.

The Cantor Set (3/3)

$x \in C$ if and only if $x = \frac{1}{3}y$ or $x = \frac{2}{3} + \frac{1}{3}y$ where $y \in C$.
$y \in [0, 1]$ so $x \in \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. So $C \subseteq \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.
$y \in \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ so $x \in \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.
$C \subseteq \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.
To get C, start with $[0, 1]$. Remove the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ from $[0, 1]$, leaving the two intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. Remove the middle thirds of those intervals, $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, leaving the four intervals $\left[0, \frac{1}{9}\right]$, $\left[\frac{2}{9}, \frac{1}{3}\right]$, $\left[\frac{2}{3}, \frac{7}{9}\right]$ and $\left[\frac{8}{9}, 1\right]$. Remove the middle thirds of those intervals, ... What's left is C, a.k.a. the Cantor Middle Thirds Set.
f is a function from $\wp(\mathbf{N})$ to $\mathbf{R}$. $y \in C$ iff $y = f(A)$ for some $A \in \wp(\mathbf{N})$. Define g from $\wp(\mathbf{N})$ to C by $g(A) = f(A)$. g is a surjection. It's also an injection, as proved in the notes. So it's a bijection and $\#\wp(\mathbf{N}) = \#C$. $\#\mathbf{N} < \#\wp(\mathbf{N})$ so $\#\mathbf{N} < \#C$. C is not countable. $C \subseteq \mathbf{R}$, so $\mathbf{R}$ is also uncountable.

Proofs of Proposition 2.7.1 (1/4)

Suppose X and Y are sets. There is an injection $f\colon X \to Y$ or there is an injection $g\colon Y \to X$.

Consider the following statements on a set $A \subseteq X \times Y$:

1. For each x in X there is at most one $y \in Y$ such that $(x, y) \in A$.
2. For each y in Y there is at most one $x \in X$ such that $(x, y) \in A$.
3. For each x in X there is at least one $y \in Y$ such that $(x, y) \in A$.
4. For each y in Y there is at least one $x \in X$ such that $(x, y) \in A$.

If A satisfies (1) and (3) we can define $f(x)$ to be the unique y such that $(x, y) \in A$. This f is injective if A satisfies (2) as well. If A satisfies (2) and (4) we can define $g(y)$ to be the unique x such that $(x, y) \in A$. This g is injective if A satisfies (1) as well.

Proofs of Proposition 2.7.1 (2/4)

A is the graph of f, or nearly the graph of g. One strategy for constructing it is to start with an A satisfying (1) and (2) and add points, without violating those, until (3) or (4) is satisfied.

1. For each x in X there is at most one $y \in Y$ such that $(x, y) \in A$.
2. For each y in Y there is at most one $x \in X$ such that $(x, y) \in A$.
3. For each x in X there is at least one $y \in Y$ such that $(x, y) \in A$.
4. For each y in Y there is at least one $x \in X$ such that $(x, y) \in A$.

$\varnothing$ satisfies (1) and (2). If we have an A satisfying (1) and (2) we can safely add (x, y) to it as long as there is no $p \in X$ such that $(p, y) \in A$ or $q \in Y$ such that $(x, q) \in A$. If we can't find such an (x, y) then either (3) or (4) is satisfied.

Proofs of Proposition 2.7.1 (3/4)

If X and Y are finite then so is $X \times Y$ and this process stops after finitely many steps.
If X and Y are infinite then the process never stops.
If X and Y are countable then so is $X \times Y$ and if we visit pairs (x, y) in order then we can be sure of visiting all of them. At no finite stage does our A satisfy (3) or (4), but the union of all of them does, because every pair gets considered and if it wasn't added then that was because we already had a (p, y) or an (x, q) in A. Taking the union is fine, since the union of any increasing sequence of sets satisfying (1) and (2) still satisfies (1) and (2). Note that if you didn't order them then the union might not satisfy (3) or (4). You could visit infinitely many but still not visit all of them.

Proofs of Proposition 2.7.1 (4/4)

It's mainly the uncountable case that we need Zorn's lemma for. S is defined to be the set of all $Z \subseteq X \times Y$ such that

1. For each x in X there is at most one $y \in Y$ such that $(x, y) \in Z$.
2. For each y in Y there is at most one $x \in X$ such that $(x, y) \in Z$.

So in our previous constructions the A at each stage was an element of S. If R is a totally ordered subset of S with respect to the inclusion relation then R has an upper bound, namely $B = \bigcup_{Z \in R} Z$. This is in S because the union of any totally ordered set of sets satisfying (1) and (2) still satisfies (1) and (2). Zorn's Lemma promises us a maximal element M of S. If you add a pair (x, y) to M then you get a larger set, so that set can't be in S. Adding it must cause (1) or (2) to fail. A Z to which you can't add any new pair satisfies either (3) or (4).

When to use Zorn's Lemma

Zorn is likely to be useful when you want to construct something,

- there's a step by step procedure which terminates after finitely many steps when the problem is of small size,
- the same procedure works for somewhat larger size, because you know how to combine the results of all the steps, but
- it doesn't work in general, because even countably many steps may not be enough.

The partially ordered set corresponds, roughly, to the possible intermediate stages of the construction. The upper bound corresponds to the way of combining them.

Notions of size depend on the situation. In our example above it was the cardinality of the sets. When proving the existence of a basis for a vector space it's the dimension.

Maximal elements aren't unique in general. If you suspect the there is only one of the thing you're constructing then probably Zorn's Lemma is not the easiest way to construct it.