# MAU22200 Lecture 11

John Stalker

Trinity College Dublin

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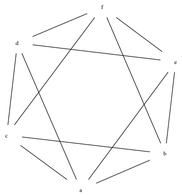
# Stronger, weaker and related concepts

It's possible to compare equivalence relations. $\sim$ is stronger than $\bowtie$ if $x \sim y$ implies $x \bowtie y$ and $\sim$ is weaker than $\bowtie$ if $x \bowtie y$ implies $x \sim y$. The terminology is a bit weird because every equivalence relation is both stronger and weaker than itself. $=$ is the strongest relation on any set. The relation $\bowtie$ where $x \bowtie y$ for all $x$ and $y$ is the weakest relation. Actually you can define stronger and weaker for any relations, not just equivalence relation. Sometimes we want to construct an equivalence relation from a relation. There's a standard way to do this. For any relation there is an equivalence relation which is weaker than it, but stronger than any other equivalence relation weaker than it.

# An example: Graph Theory

A natural example of that construction comes from Graph Theory. These are graphs with nodes and edges, not graphs of functions. In other words, things like




"is connected to by an edge" is a relation on the set of nodes which satisfies the second condition to be an equivalence relation, but not the other two. "can be connected to by a finite sequence of edges" is an equivalence relation. It's weaker than "is connected to by an edge" but stronger than an other equivalence relation weaker than it.

# Cantor's Theorem

For any set $X$ the function $f\colon X \to \wp(X)$ defined by $f(x) = \{x\}$ is an injection. It is never a surjection. $\varnothing$ is not in its image, for example. Is there some other $f\colon X \to \wp(X)$ which is a surjection? No.
For finite sets this is obvious. If $\#X = n$ then $\#\wp(X) = 2^n$. $2^n > n$ so there aren't enough elements of $X$ to map onto all the elements of $\wp(X)$.
Cantor's Theorem extends this to (possibly) infinite sets. The proof is surprisingly short, and the result is surprisingly useful. If you look carefully at the proof in the notes you can find an echo of the argument for why there is no set of all sets. Given a surjection $f\colon X \to \wp(X)$ we set $A = \{y \in X : y \notin f(y)\}$. If there's an $x \in X$ such that $f(x) = A$ then ask whether $x \in A$. If it is then it isn't and vice versa.

# The Schröder-Bernstein Theorem (1/2)

We're working towards a theory of cardinalities of sets. Cantor's Theorem will ultimately say that $X$ is of strictly lower cardinality than $\wp(X)$. The Schröder-Bernstein theorem will say the if the cardinality of $X$ is no larger than that of $Y$ and the cardinality of $Y$ is no larger than that of $X$ then they are of equal cardinality. But I haven't defined cardinal numbers yet. The version without cardinalities is this

> *Suppose $f\colon X \to Y$ and $g\colon Y \to X$ are injections. Then there is a bijection $h\colon X \to Y$.*

The proof is fairly long, but the idea isn't too complicated. It would be easier if we assumed that $X \cap Y = \varnothing$. The first step is therefore to construct a disjoint union of $X$ and $Y$. Here we only need the disjoint union of two sets but this is just a simple case of a more general construction.

# The Schröder-Bernstein Theorem (2/2)

The main idea is to label each element of the disjoint union of $X$ and $Y$ with a label to indicate whether it belongs to the copy of $X$ or to the copy of $Y$. $X$ and $Y$ themselves aren't suitable as labels because we can't guarantee $X \neq Y$. 1 and 2 would work but I chose $a$ and $b$.

The main idea of the proof is to form a graph whose nodes are the elements of the disjoint union and where the edges connect $(a, x)$ to $(b, f(x))$ and $(b, y)$ to $(a, g(y))$. The connected components are of four types: closed loops, singly linked chains stretching out infinitely in both directions, singly linked chains extending infinitely from an element labelled $a$, and singly linked chains extending infinitely from an element labelled $b$. On each connected component you shift elements either one step forward or one step backward along the chain.

# Zorn's Lemma

The "C" in "ZFC" refers to the Axiom of Choice. Roughly, it says that given non-empty set valued function there is a function which chooses one element from each set. You can use this, for example, to show that every surjection has a right inverse. If $f : X \to Y$ what you need is a function $g$ taking each $y \in Y$ to an element of $f^*(\{y\})$.

"Zorn's Lemma" was first proved by Kuratowski as a consequence of the Axiom of Choice. Zorn suggested taking the lemma as an axiom and proving the former axiom as a consequence of it, but he never actually did this.

It's a good idea though because it's usually easier to use Zorn's Lemma than the Axiom of Choice.

To state Zorn we'll need some terminology about partial orders.

# Partial orders

A *partial order* is a relation $\preccurlyeq$ satisfying

- $a \preccurlyeq a$.
- If $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$.
- If $a \preccurlyeq b$ and $b \preccurlyeq a$ then $a = b$.

A set with a partial order is a *partially ordered set*. This is not the same as a directed set. The third condition is different.
On a partially ordered set you can define the notions of maximal, greatest, and upper bound, and the dual notions of minimal, least and lower bound. Maximal and greatest are not the same!
Greatest means greater then everything else. Maximal means nothing else is strictly greater. For example, the set of non-empty proper subsets of a given set, ordered by inclusion, has minimal elements, the one element sets, and maximal elements, the all but one element sets, but no greatest or least element.

# Zorn again

Upper and lower bounds are always relative to a subset, e.g. an upper bound for $R$ in $S$. That's an element of $S$ which is at least as great as any element of $R$. If we required it to be in $R$ it would be a greatest element. For example, 1 is an upper bound for $(0, 1)$ in $\mathbf{R}$, but not a greatest element because it's not in $(0, 1)$.
We can now state Zorn:

> *Suppose $(S, \preccurlyeq)$ is a partially ordered set. Suppose that for each $R \in \wp(S)$ such that the restriction of $\preccurlyeq$ to $R$ is a total order on $R$, the set $R$ has an upper bound. Then $S$ has a maximal element.*

This is fairly opaque, but very useful!
It's used several times in this chapter and will be used again. One application is that for any two sets there's an injection from one of them to the other. It takes a bit of practice to learn how to use Zorn's Lemma.

# Cardinal numbers

We use $\#X = \#Y$ as a short hand for "there is a bijection from $X$ to $Y$" and $\#X \leq \#Y$ as a short hand for "there is an injection from $X$ to $Y$". I will not attempt to attach a meaning to $\#X$ separately, except when $X$ is a finite set. It can be done, but we don't need it. There's also $\#X \geq \#Y$ as an alternate notation for $\#X \leq \#Y$.

This wouldn't be useful if the symbols $=$ and $\leq$ didn't behave much as they do for natural numbers. For example, if $\#X \leq \# \ Y$ and $\#Y \leq \#X$ then $\#X = \#Y$ is the Schröder-Bernstein theorem. If $\#X \leq \#Y$ and $\#Y \leq \#Z$ then $\#X \leq \#Z$. This is the fact that the composition of injections is an injection. There's a long list of properties in the notes. It would be even longer if I'd defined $<$ and $>$.