MAU22200 Lecture 10

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4 October 2021

Set Theory is weird (1/4)

Set theory is weird. There are several versions of it, but there is one commonly used in the rest of Mathematics, known as ZFC. ZFC mostly matches your intuitive ideas about sets, but not fully. Intuitively properties define sets and the set operations of intersection, union and complement correspond to the logical operations of *and*, *or* and *not*. If A is the set of x such that $P(x)$ where P is some property and B is the set of x such that $Q(x)$ for some other property then $A \cap B$ is the set of x such that $P(x)$ *and* $Q(x)$. For example if P is the property of being an SF student and Q is the property of being a single honours Maths student then A is the set of SF students, B is set of single honours Maths students and $A \cap B$ the set of SF single honours Maths students, i.e. all x such that x is an SF student *and* x is single honours Maths student.

Set Theory is weird (2/4)

Similarly $A \cup B$ is the set of all x such that x is an SF student *or* x is single honours Maths student, i.e. the set of all students who are either SF or single honours Maths.

A^c, the complement of A would be the set of x such that x is not an SF student. Note this isn't the same as the set of all students who are not SF students. Neptune, for example, would be an element of A^c, because it, like the other planets, is not an SF student. If you want the set of students who are not SF students then what you need is the relative complement $S \setminus A$, where S is the set of students.

The set theory I've just described is the one where people draw Venn diagrams. It is not ZFC. ZFC does not have complements! It does have relative complements, though.

Set Theory is weird (3/4)

Why doesn't ZFC have complements? You could use them to construct the set of all sets, via $A \cup A^c$. The set of all sets would have a subset $\mathcal{R}$, consisting of all sets S such that $S \notin S$. Is $\mathcal{R} \in \mathcal{R}$? If it is then $\mathcal{R} \notin \mathcal{R}$, so it isn't. If it isn't then $\mathcal{R} \in \mathcal{R}$, so it is. This is not a logically consistent theory.

ZFC is designed to be a logically consistent theory rather than an intuitive one. Does it succeed in being logically consistent? No one has shown that it isn't. No one can show that it is.

There are other versions of Set Theory which avoid the paradox above in other ways. Some have complements and others don't. In some the existence of complements of sets is undecidable. I'm not going to talk about any of them. Mathematicians who aren't set theorists or logicians mostly use ZFC.

Set Theory is weird (4/4)

You are the empty set!
The statement above may surprise you but it follows immediately from the first axiom of set theory, the Axiom of Extensionality.

$$\forall x \colon \forall y \colon \; (\forall z \colon \; (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)\,.$$

This says that x and y are equal if they have the same elements, i.e. if z is an element of x if and only if it is any element of y.
The empty set has no elements by definition. You have no elements because you are not a set. Therefore you are, by the Axiom of Extensionality, equal to the empty set. Don't take it personally, so is everyone else.
ZFC does not describe the universe we live in and was never intended to. It was intended as a foundation for Mathematics. Even as that, it's a bit odd. The intuition you get by thinking about "sets" like the "set" of SF students or the "set" of SH Maths students doesn't fully apply.

Chapter 2

Chapter 2 of the notes is about Set Theory. Its purpose is to teach you as little of Set Theory as possible, i.e. just enough that you can get through the undergraduate curriculum.
For purposes of this module the main point is towards the end, where I discuss cardinality and countable sets. Almost everything else is there in order to be able to state and prove those results. Some of it will be needed throughout the module and some is really only there for completeness.
You don't need to remember any of the proofs, although some contain ideas that we will use again. You do need to remember, or be able to reconstruct, nearly all the statements.

Injections, surjections and bijections

These notions are used everywhere in Mathematics. The adjective forms injective, surjective and bijective are also used. Some authors, especially older British ones, use the term "one-to-one" and "onto" for injective and surjective, and "one-to-one and onto" for bijective.

These are closely connected with the notions of left inverses, right inverses and inverses. A left inverse to a function f is a function g such that $g \circ f$ is the identity function. A right inverse to a function f is a function g such that $f \circ g$ is the identity function. Note that these are, in general, different identity functions! If $f : X \to Y$ then $g \circ f$ is the identity on X and$f \circ g$ is the identity on Y. An inverse is a function which is both a left inverse and a right inverse.

Injectivity, surjectivity and bijectivity are more or less equivalent to the existence of a left inverse, right inverse and inverse, respectively.

Subtleties

Injectivity, surjectivity and bijectivity are more or less equivalent to the existence of a left inverse, right inverse and inverse, respectively.

What do I mean by "more or less"? If X is non-empty then $f : X \to Y$ is an injection if and only if it has a left inverse. If X is empty then there is a (vacuous) function from X to Y for any Y and it is (vacuously) an injection. It has no left inverse unless Y is also empty, because there is no function from a non-empty set to an empty one.

$f : X \to Y$ is a surjection if and only if it has a right inverse. The "if" is straightforward but the "only if" is more subtle. It's related to the "C" in "ZFC", which stands for "Choice". If $f : X \to Y$ is a surjection then there is for each $y \in Y$ an $x \in X$ such that $f(x) = y$. If there were a unique such x we could say "let $g(y)$ be the unique $x \in X$ such that $f(x) = y$". Can we say "let $g(y)$ be *some* $x \in X$ such that $f(x) = y$"?

Finite sets

A proper module on the foundations of Mathematics would proceed in the following order: Logic, Set Theory, construction of the natural numbers, construction of the integers, construction of the rational numbers, construction of the real numbers, construction of the complex numbers.
This is not a module on the foundations of Mathematics and I won't do this, but some definitions reflect the path above. In particular, the integers are (or would be) constructed using Set Theory, so definitions in Set Theory shouldn't refer to any of the various types of numbers. That's why the definition of finite sets in the notes is weird. I define a set as finite if every injection from the set to itself is a surjection.
We won't fully pretend we've never seen the natural numbers though. I use them in examples and there's an alternate characterisation of infinite sets as those sets X for which there is an injection $f\colon \mathbf{N} \to X$.

Equivalence relations, equivalence classes

An equivalence relation is a relation $\sim$ satisfying

- $x \sim x$
- if $x \sim y$ then $y \sim x$
- if $x \sim y$ and $y \sim z$ then $x \sim z$.

An equivalence relation on a set partitions the set into equivalence classes. Each element belongs to exactly one equivalence class. Many useful objects are constructed as equivalence classes. The usual constructions of the integers, rationals and reals are via equivalence classes. Some constructions of the natural numbers and complex numbers are via equivalence classes. I won't give any of those constructions, but we will see equivalence classes later.