MAU22200 Lecture 9

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30 September 2021

Neighbourhoods

The properties of the set of neighbourhoods were as follows

1. $\mathcal{N}(w) \neq \varnothing$
2. $\varnothing \notin \mathcal{N}(w)$
3. For any $A, B \in \mathcal{N}(w)$ there's a $C \in \mathcal{N}(w)$ such that $A \supseteq C$ and $B \supseteq C$.
4. If $A \in \mathcal{N}(w)$ and $A \subseteq B$ then $B \in \mathcal{N}(w)$.

$\mathcal{O}(w)$ has the same properties, except for the last. We can now recognise the third property as the statement that $(\mathcal{N}(w), \supseteq)$ is a directed set.

We want a name for things which behave like $\mathcal{N}(w)$ or $\mathcal{O}(w)$.

Filters and prefilters

We say that $\mathcal{F} \in \wp(\wp(X))$ is a *filter* on X if

1. $\mathcal{F} \neq \varnothing$
2. $\varnothing \notin \mathcal{F}$
3. For any $A, B \in \mathcal{F}$ there's a $C \in \mathcal{F}$ such that $A \supseteq C$ and $B \supseteq C$.
4. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

$\mathcal{F}$ is called a prefilter if it satisfies the first three. $\mathcal{N}(w)$ is a filter (and a prefilter) while $\mathcal{O}(w)$ is a prefilter (but not usually a filter). In our general setting for discussing limits, the set $\omega_*(D)$ satisfies the first and third conditions. To get a sensible theory of limits we want it to satisfy the second as well. In other words, we want it to be a prefilter.

Extra hypotheses

Our general setting involves a domain U, a co-domain Y with a topology $\mathcal{T}$, a function f from U to Y, a non-empty directed set $(D, \preccurlyeq)$, and a monotone function ω from $(D, \preccurlyeq)$ to $(\wp(U), \supseteq)$.
We'd like $\mathcal{F} = \omega_*(D)$ to be a prefilter. It's the image of a filter under a monotone function so $\mathcal{F} \neq \varnothing$ and for any $A, B \in \mathcal{F}$ there's a $C \in \mathcal{F}$ such that $A \supseteq C$ and $B \supseteq C$. The missing condition is $\varnothing \notin \mathcal{F}$.
For $\lim_{x \to w} f(x)$ we had $w \in U \subseteq X$, $(D, \preccurlyeq) = (\mathcal{O}(w), \supseteq)$, and $\omega(W) = U \cap W \setminus \{w\}$. $\omega_*(D)$ is the set of subsets of U of the form $U \cap W \setminus \{w\}$ for some $W \in \mathcal{O}(w)$. So $\varnothing \notin \omega_*(D)$ means that for every $W \in \mathcal{O}(w)$ the set $U \cap W \setminus \{w\}$ is non-empty.
That's precisely the condition for w to be a limit point of U.

Extra hypotheses, continued

For $\lim_{x \to +\infty} f(x)$ we had $U \subseteq \mathbf{R}$, $(D, \preccurlyeq) = (\mathbf{R}, \leq)$, and $\omega(a) = U \cap [a, +\infty)$. $\omega_*(D)$ is the set of subsets of U of the form $U \cap [a, +\infty)$ for some $a \in \mathbf{R}$. So $\varnothing \notin \omega_*(D)$ means that for every $a \in \mathbf{R}$ the set $U \cap [a, +\infty)$ is non-empty. In other words, for every $a \in \mathbf{R}$ there's an $x \in U$ with $x \geq a$. Or, equivalently, U has no upper bound. We should expect that $\lim_{x \to +\infty} f(x)$ is unique under that condition.

For $\lim_{n \to \infty} \alpha_n$ we had $U = \mathbf{N}$, $(D, \preccurlyeq) = (\mathbf{N}, \leq)$, and $\omega(m) = \{n \in \mathbf{N} \colon m \leq n\}$. $\omega_*(D)$ is the set of subsets of $\mathbf{N}$ of the form $\{n \in \mathbf{N} \colon m \leq n\}$ for some $m \in \mathbf{N}$. So $\varnothing \notin \omega_*(D)$ means that for every $m \in \mathbf{N}$ the set $\{n \in \mathbf{N} \colon m \leq n\}$ is non-empty. This is true. We should expect that $\lim_{n \to \infty} \alpha_n$ is unique without any extra hypotheses.

You should be able to figure out what hypothesis we need to ensure that $\lim_{x \searrow w} f(x)$ is unique.

Limits in general

Suppose U, Y are sets, $\mathcal{T}$ is a topology on Y, f is a function from U to Y, $(D, \preccurlyeq)$ is a non-empty directed set, and ω is a monotone function from $(D, \preccurlyeq)$ to $(\wp(U), \supseteq)$. We say that $z \in Y$ is the limit of f with respect to ω, written

$$\lim_{\omega} f = z,$$

if for all $P \in \mathcal{N}(z)$ there's an $a \in D$ such that if $x \in \omega(a)$ then $f(x) \in P$.

We can prove analogues of the theorems for $\lim_{x \to w} f(x)$ for $\lim_{\omega} f$, e.g.

> *If $\varnothing \notin \omega_*(D)$ and $\mathcal{T}$ is Hausdorff then there is at most one $z \in Y$ such that $\lim_{\omega} f = z$.*

There's also a monotonicity theorem if $Y = \mathbf{R}$ and a linearity theorem if Y is a normed vector space.

Sums

We can develop a general theory of sums along the same lines. Series are limits of partial sums, specifically the limit of the sum of the first n summands. In general we have no concept of "first n" elements, even for sets like $\mathbf{Q}$ which can be put in 1-to-1 correspondence with the natural numbers. We still have a concept of finite though.

Our set U is the set F of finite subsets of S. Our set D is also F and our relation $\preccurlyeq$ is $\subseteq$. Our ω takes a finite subset A to the set of all finite sets containing A. Our $f \colon U \to Y$ takes a finite subset to the sum over that subset. This makes sense in a context where we can add pairs of elements, e.g. if Y is a normed vector space. The hypothesis $\varnothing \notin \omega_*(D)$ is always satisfied. We can define the sum over S to be the limit f with respect to ω.

Most, but not all, of the basic properties of sums follow from those of limits.

Integrals

It's also possible to express integrals as limits. In fact that's the main reason we're doing this. It will make this semester harder, but next semester much easier.

It's not obvious how to do this, and there's more than one option. The usual theory of Riemann integration involves partitioning an interval into finitely many subintervals. There's a notion of refinement and common refinement. It make sense to choose D to be the set of partitions of the interval of integration and choose $\preccurlyeq$ to be the "is refined by" relation. The existence of a common refinement shows that $(D, \preccurlyeq)$ is a directed set. It's non-empty because there's always the trivial partition.

We also need a U. Riemann's construction of Riemann integral involves choosing a point in each subinterval and multiplying by the length of the subinterval. It's equivalent, and better for our purposes, to choose finitely many points and give each a weight, such that the weights add up to the length of the subinterval.

Integrals, continued

We can choose U to be the set of systems of weights, i.e. finite sets of points each of which is assigned a non-negative weight. The function $\omega \colon D \to \wp(U)$ takes each partition to the set of systems of weights compatible with it, i.e. such that the length of each subinterval is the sum of the weights of the points it contains. It's monotone. There's always at least one compatible system of weights so $\omega_*(D)$ is non-empty.
We also need a function $f \colon U \to \mathbf{R}$ (or $f \colon U \to Y$, for suitable Y). The sensible choice of f takes a system of weights to the weighted sum of the integrand, i.e. the sum of the value at each point in the interval times the weight of that point.
The limit of this f with respect to this ω is then the Riemann integral of the integrand over the interval. It automatically has (most of) the properties we expect, e.g uniqueness, monotonicity, linearity, etc. because limits have these properties.