MAU22200 Lecture 8

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Directed sets

Recall that a directed set is a pair $(D, \preccurlyeq)$ where $\preccurlyeq$ is a relation on D satisfying

- $a \preccurlyeq a$.
- If $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$.
- For any a, b there is a c such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

The last property is stated for pairs, but applies to finite collections in general. For any $a_1, a_2, \ldots, a_k$ there is a c such that $a_1 \preccurlyeq c$, $a_2 \preccurlyeq c$, $\ldots$, $a_k \preccurlyeq c$.
The directed sets relevant to the limits $\lim_{x\to w} f(x)$, $\lim_{x\to+\infty} f(x)$, $\lim_{n\to\infty} \alpha_n$, and $\lim_{x\searrow w} f(x)$ are $(\mathcal{O}(w), \supseteq)$, $(\mathbf{R}, \leq)$, $(\mathbf{N}, \leq)$, and $((0, +\infty), \geq)$ respectively.

Directed set examples (1/3)

The requirements are very weak, so there are many directed sets. The conditions that $a \preccurlyeq a$ and if $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$ are obvious in each of the cases below and so will be ignored.

- The set N of non-empty subsets of a given set with the relation $\subseteq$. If A, B are non-empty then there's a non-empty C such that $A \subseteq C$ and $B \subseteq C$. $C = A \cup B$ works.
- The set P of proper subsets of a given set with the relation $\supseteq$. If A, B are non-empty then there's a proper C such that $A \supseteq C$ and $B \supseteq C$. $C = A \cap B$ works. Note that $(P, \subseteq)$ isn't a directed set. If $C = A \cup B$ then $A \subseteq C$ and $B \subseteq C$, but C needn't be proper. Similarly, $(N, \supseteq)$ isn't a directed set.
- The set V of finite dimensional subspaces of a given vector space with the relation $\supseteq$ is a directed set. If A, B are finite dimensional subspaces then so is $C = A \cap B$ and $A \supseteq C$ and $B \supseteq C$.

Directed set examples (2/3)

- The set V of finite dimensional subspaces of a given vector space with the relation $\subseteq$ is a directed set. We cannot use $A \cup B$ in this case. $A \subseteq A \cup B$ and $B \subseteq A \cup B$, but $A \cup B$ is generally not a subspace. The correct choice of C is $C = A + B$. $A \subseteq A + B$ and $B \subseteq A + B$, and $A + B$ is a finite dimensional subspace.
- The set F of finite subsets of a given set with the relation $\subseteq$ is a directed set. If A, B are finite subsets then there's a finite subset C such that $A \subseteq C$ and $B \subseteq C$. $C = A \cup B$ works.

Directed set examples (3/3)

- The set $\mathcal{B}(x)$ of open balls containing a given point x in a metric space (X, d), with the relation $\subseteq$. If $x \in B(y, r)$ and $x \in B(z, s)$ then $B(y, r) \subseteq B(x, \max(2r, 2s))$ and $B(z, s) \subseteq B(x, \max(2r, 2s))$. This is an easy consequence of the triangle inequality.
- The set $\mathcal{B}$ of all open balls with the relation $\subseteq$. $B(y, r) \subseteq B(y, \max(r, d(y, z) + s))$ and $B(z, s) \subseteq B(z, \max(r, d(y, z) + s))$.
- The set $B(x)$ with the relation $\supseteq$. If $x \in B(y, r)$ and $x \in B(z, s)$ then $B(y, r) \supseteq B(x, r - d(x, y), s - d(x, z))$ and $B(z, s) \supseteq B(x, r - d(x, y), s - d(x, z))$. $\mathcal{B}$ with the relation $\supseteq$ is not generally a directed set.

Directed sets and limits

Remember the table from last lecture:

$f\colon U \to Y$	$\lim_{x\to w}$	$W \in \mathcal{O}(w)$	$x \in U \cap W \setminus \{w\}$
$f\colon U \to Y$	$\lim_{x\to +\infty}$	$a \in \mathbf{R}$	$x \in U$ and $x \geq a$
$\alpha\colon \mathbf{N} \to Y$	$\lim_{n\to\infty}$	$m \in \mathbf{N}$	$n \geq m$
$f\colon U \to Y$	$\lim_{x\searrow w}$	$\delta > 0$	$w < x < w + \delta$

The third column features a parameter, which belongs to a non-empty directed set. The fourth column features a subset of the domain of our function/sequence, which depends on this parameter. The sets are $U \cap W \setminus \{w\}$, $U \cap [a, +\infty)$, $\{n \in \mathbf{N}\colon m \leq n\}$, and $(w, w+\delta)$ respectively. In each case the sets of that form, as the parameter varies, form a directed set with the relation $\supseteq$. For example, given $W_1, W_2 \in \mathcal{O}(w)$, there is a $W \in \mathcal{O}(w)$ such that $U \cap W_1 \setminus \{w\} \supseteq U \cap W \setminus \{w\}$ and $U \cap W_2 \setminus \{w\} \supseteq U \cap W \setminus \{w\}$. $W = W_1 \cap W_2$ works.

Monotone functions

Both the third and fourth columns feature a directed set, and there's a consistent relation between the two.

If $(D, \preccurlyeq)$ and $(E, \preccurlyeq)$ are directed sets then a function ω from D to E is called *monotone* if $\omega(a) \preccurlyeq \omega(b)$ whenever $a \preccurlyeq b$.

The function from $(\mathcal{O}(w), \supseteq)$ to $(\wp(U), \supseteq)$ which takes W to $U \cap W \setminus \{w\}$ is monotone. In other words, $U \cap W_1 \setminus \{w\} \supseteq U \cap W_2 \setminus \{w\}$ whenever $W_1 \supseteq W_2$.

Similarly, the function from $(\mathbf{R}, \leq)$ to $(\wp(U), \supseteq)$ which takes a to $U \cap [a, +\infty)$ is monotone. $U \cap [a, +\infty) \supseteq U \cap [b, +\infty)$ whenever $a \leq b$.

You can check that the functions for the other two rows are also monotone. The sets in the fourth column are the images of these functions from the directed set in the third column to the power set of the domain. Images of non-empty directed sets under monotone functions are non-empty directed sets, which is why the sets in the fourth column are non-empty directed sets.

That table, again

Here's that table again:

$f: U \to Y$	$\lim_{x\to w}$	$W \in \mathcal{O}(w)$	$x \in U \cap W \setminus \{w\}$
$f: U \to Y$	$\lim_{x\to +\infty}$	$a \in \mathbf{R}$	$x \in U$ and $x \geq a$
$\alpha: \mathbf{N} \to Y$	$\lim_{n\to\infty}$	$m \in \mathbf{N}$	$n \geq m$
$f: U \to Y$	$\lim_{x\searrow w}$	$\delta > 0$	$w < x < w + \delta$

Here's a new table reflecting what we've just observed:

function	limit	directed set	monotone function
$f: U \to Y$	$\lim_{x\to w}$	$(\mathcal{O}(w), \supseteq)$	$\omega(W) = U \cap W \setminus \{w\}$
$f: U \to Y$	$\lim_{x\to +\infty}$	$(\mathbf{R}, \leq)$	$\omega(a) = U \cap [a, +\infty)$
$\alpha: \mathbf{N} \to Y$	$\lim_{n\to\infty}$	$(\mathbf{N}, \leq)$	$\omega(m) = \{n \in \mathbf{N} : m \leq n\}$
$f: U \to Y$	$\lim_{x\searrow w}$	$((0, +\infty), \geq)$	$\omega(\delta) = (w, w + \delta)$

The general setting

We now have most of what we need to describe a general theory of limits. The ingredients are:

- a domain U
- a co-domain Y with a topology $\mathcal{T}$
- a function f from U to Y
- a non-empty directed set $(D, \preccurlyeq)$
- a monotone function ω from $(D, \preccurlyeq)$ to $(\wp(U), \supseteq)$

The image $\omega_*(D)$ with the relation $\supseteq$ is a non-empty directed set. To get a sensible theory of limits we need it to have one more property.