MAU22200 Lecture 7

John Stalker

Trinity College Dublin

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Neighbourhoods

If $(X, \mathcal{T})$ is a topological space and $w \in X$ then $\mathcal{N}(w)$ is the set of neighbourhoods of w. In other words, $V \in \mathcal{N}(w)$ if and only if there is a $U \in \mathcal{T}$ such that $w \in U$ and $U \subseteq V$.
$\mathcal{N}(w)$ has the following properties:

1. $\mathcal{N}(w) \neq \varnothing$ because $X \in \mathcal{N}(w)$.
2. $\varnothing \notin \mathcal{N}(w)$ because $V \in \mathcal{N}(w) \Rightarrow w \in V$.
3. For any $A, B \in \mathcal{N}(w)$ there's a $C \in \mathcal{N}(w)$ such that $A \supseteq C$ and $B \supseteq C$. This one is less obvious, but follows from the corresponding property of $\mathcal{T}$.
4. If $A \in \mathcal{N}(w)$ and $A \subseteq B$ then $B \in \mathcal{N}(w)$.

The first three hold for the set $\mathcal{O}(w)$ of open neighbourhoods of w. The fourth usually doesn't.

Where are we going? (1/n)

We have enough to state and prove the following theorem (Theorem 1.17.1):

> *Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, that $U \in \wp(X)$ and that $w \in X$ is a limit point of U.*
>
> *If f is a function from U to Y and $\mathcal{T}_Y$ is Hausdorff then there is at most one $z \in Y$ such that $\lim_{x \to w} f(x) = z$.*
>
> *Suppose $Y = \mathbf{R}$ and $\mathcal{T}$ is the usual topology on $\mathbf{R}$. If f and g are functions from U to Y such that $f(x) \leq g(x)$ for all $x \in X$ and $\lim_{x \to w} f(x)$ and $\lim_{x \to w} g(x)$ exist then $\lim_{x \to w} f(x) \leq \lim_{x \to w} g(x)$.*

Where are we going? (2/n)

Suppose Y is a vector space and $\mathcal{T}$ is the topology associated to a norm q on Y. Suppose

$$\mathbf{g} = \sum_{j=1}^{k} \alpha_j \mathbf{f}_j$$

where $\mathbf{f}_1, \ldots \mathbf{f}_k$ are functions from U to Y, $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$, and

$$\lim_{x \to w} \mathbf{f}_j(x) = \mathbf{z}_j,$$

where $\mathbf{z}_1, \ldots \mathbf{z}_k \in Y$. Then

$$\lim_{x \to w} \mathbf{g}(x) = \sum_{j=1}^{k} \alpha_j \mathbf{z}_j.$$

Where are we going? (3/n)

The limit can be defined in terms of neighbourhoods (Proposition 1.13.3):

> $\lim_{x \to w} f(x) = z$ *if and only if for all* $T \in \mathcal{N}(z)$ *there is a* $W \in \mathcal{O}(w)$ *such that if* $x \in U \cap W \setminus \{w\}$ *then* $f(x) \in T$*.*

There are a number of other types of limits with similar looking definitions.

> $\lim_{x \to +\infty} f(x) = z$ *if and only if for all* $Z \in \mathcal{O}(z)$ *there is an* $a \in \mathbf{R}$ *such that if* $x \in U$ *and* $x \geq a$ *then* $f(x) \in Z$*.*

In the special case $U = \mathbf{N}$ this is just the limit of a sequence. For sequences people usually use subscript notation rather than functional, i.e. α_n rather than $f(n)$.

> $\lim_{n \to \infty} \alpha_n = z$ *if and only if for all* $Z \in \mathcal{O}(z)$ *there is an* $m \in \mathbf{N}$ *such that if* $n \geq m$ *then* $\alpha_n \in Z$*.*

Where are we going? (4/n)

> $\lim_{x \searrow w} f(x) = z$ *if and only if for all* $Z \in \mathcal{O}(z)$ *there is a* $\delta > 0$ *such that if* $x \in U$ *and* $w < x < w + \delta$ *then* $f(x) \in Z$.

I happened to state these in terms of $\mathcal{O}(z)$ and the other in terms of $\mathcal{N}(z)$, but that doesn't really matter. Theorem 1.16.2 gives 14 equivalent versions of any of these definitions.

Each of these types of limit has a similar theorem to the one for $\lim_{x \to w}$. There's usually some extra hypothesis, analogous to the hypothesis that w is a limit point of U. In the case of $\lim_{x \to +\infty}$ it's the set U has no upper bound. We'd like a single theorem from which we derive all of these and more. The hypotheses of that theorem can help us identify which is the correct extra hypothesis in each case.

Where are we going? (5/n)

Let's construct a table to illustrate the analogies.

$f : U \to Y$	$\lim_{x \to w}$	$W \in \mathcal{O}(w)$	$x \in U \cap W \setminus \{w\}$
$f : U \to Y$	$\lim_{x \to +\infty}$	$a \in \mathbf{R}$	$x \in U$ and $x \geq a$
$\alpha : \mathbf{N} \to Y$	$\lim_{n \to \infty}$	$m \in \mathbf{N}$	$n \geq m$
$f : U \to Y$	$\lim_{x \searrow w}$	$\delta > 0$	$w < x < w + \delta$

From this table you can almost fill in the definitions. Our general theorem will need a mathematical structure corresponding to the third column and one corresponding to the fourth column. These structures should have all the properties needed to make the proof work. For example, what do the sets $\mathcal{O}(w)$, $\mathbf{R}$, $\mathbf{N}$ and $(0, +\infty)$ have in common? Is it something which we can use to prove limit theorems?

Where are we going? (6/n)

There are some extra rows we want to add to this table.
We want a theory of (infinite) sums.
There's already a theory of series, i.e. sums of functions from $\mathbf{N}$ to $\mathbf{R}$ or to $\mathbf{C}$ or $\mathbf{R}^n$ or to a normed vector space. This theory uses the order structure on $\mathbf{N}$.
We want to be able to sum functions defined on sets which don't have a natural order, e.g. the sum of the volumes of a collection of balls in $\mathbf{R}^n$.
We also want a theory of integrals. In fact we want at least two, but we'll start with the Riemann integral.
Sums and integrals have similar properties to limits. Can we get them from the same theorem?

Directed sets

What do the sets $\mathcal{O}(w)$, $\mathbf{R}$, $\mathbf{N}$ and $(0, +\infty)$ have in common? They each have an order structure. We have $\subseteq$ for $\mathcal{O}(w)$ and $\leq$ for the other three. There's also the opposite order, $\supseteq$ or $\geq$. We need to choose one in each setting and this choice is dictated by the types of arguments we need. Look at the argument for limits of sums. We have a particular W_j, a_j, m_j or δ_j which controls where the values of $\mathbf{f}_j$ are sufficiently close to $\mathbf{z}_j$. For the sum we choose a W which is a subset of all of the W_j, an a which is greater (or equal to) than all the a_j, an m which is greater than all the m_j, or a δ which is less than all the δ_j.
We define a *directed set* to be a pair $(D, \preccurlyeq)$ where $\preccurlyeq$ is a relation on D satisfying

- $a \preccurlyeq a$.
- If $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$.
- For any a, b there is a c such that $a \preccurlyeq c$ and $b \preccurlyeq c$.