MAU22200 Lecture 6

John Stalker

Trinity College Dublin

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Notes

I'm posting an updated set of notes. The following changes have been made.

- Chapter 1 is now titled Limits rather than Introduction.
- Various typos have been corrected. No doubt I missed some. Let me know of any you find.
- More detail has been added to some proofs.
- I've added cross-reference links to make it easier to navigate with most pdf readers.
- There are several new sections.

Chapter 1 mostly covers things we'll revisit in more detail in later chapters. You aren't really meant to follow everything, especially in later sections. You'll see more examples, constructions, properties, etc. for each topic in later chapters.

Neighbourhoods

We often want to say something is true at all points "near" a given point. For example, if f is continuous at w and $f(w) > 0$ then $f(x) > 0$ for all x near w. Neighbourhoods make this notion of nearness precise. We'll interpret "near" as "in a neighbourhood of". If $(X, \mathcal{T})$ is a topological space, $w \in X$ and $V \subseteq X$ then V is a neighbourhood of w if and only if there is a $U \in \mathcal{T}$ such that

- $w \in U$, and
- $U \subseteq V$.

This is defined for topological spaces, but will be easier to understand in lower levels of generality.
Note that if V is a neighbourhood of w and $V \subseteq W$ then W is also a neighbourhood of w.

Neighbourhoods in a metric space

Suppose (X, d) is a metric space and $\mathcal{T}$ is the usual topology, i.e. the topology of open sets.
If $w \in U$ and $U \in \mathcal{T}$ then there is an open ball about w contained in U. In other words, there is an $r > 0$ such that $B(w, r) \subseteq U$.
From $B(w, r) \subseteq U$ and $U \subseteq V$ it follows that $B(w, r) \subseteq V$. In other words, there is an $r > 0$ such that if $d(w, x) < r$ then $x \in V$, i.e. all points whose distance from w is sufficiently small belong to V.
Conversely, suppose V is a subset of X and there is an $r > 0$ such that if $d(w, x) < r$ then $x \in V$. Then $B(w, r) \subseteq V$. $B(w, r)$ is an open set, i.e. an element of $\mathcal{T}$, and $w \in B(w, r)$, so there is a $U \in \mathcal{T}$ such that $w \in U$ and $U \subseteq V$, namely $U = B(w, r)$. So V is a neighbourhood of w
$V \subseteq X$ is a neighbourhood of $w \in X$ if and only if there is an $r > 0$ such that $x \in V$ whenever $d(w, x) < r$.

Neighbourhoods in normed vector spaces, $\mathbf{R}^n$, $\mathbf{R}$

Suppose (X, p) is a normed vector space and d is the usual metric $d(\mathbf{x}, \mathbf{y}) = p(\mathbf{x} - \mathbf{y})$. Then $d(\mathbf{w}, \mathbf{x}) < r$ if and only if $p(\mathbf{w} - \mathbf{x}) < r$. $V \in X$ is a neighbourhood of $\mathbf{w} \in X$ if and only if there is an $r > 0$ such that $\mathbf{x} \in V$ whenever $p(\mathbf{w} - \mathbf{x}) < r$.
In the particular case where $X = \mathbf{R}^n$ and p is the Euclidean norm, $V \in \mathbf{R}^n$ is a neighbourhood of $\mathbf{w} \in \mathbf{R}^n$ if and only if there is an $r > 0$ such that $\mathbf{x} \in V$ whenever $\|\mathbf{w} - \mathbf{x}\| < r$.
In the case $n = 1$, $V \subseteq \mathbf{R}$ is a neighbourhood of $w \in \mathbf{R}$ if and only if there is an $r > 0$ such that $x \in V$ whenever $|w - x| < r$. Equivalently, whenever $-r < w - x < r$, or whenever $w - r < x < w + r$, or whenever $x \in (w - r, w + r)$.
If $V \subseteq \mathbf{R}$ is a neighbourhood of $w \in \mathbf{R}$ then there is an open interval (a, b) such that $a < w < b$ and $(a, b) \subseteq V$. Conversely, if there is such an (a, b) then V is a neighbourhood of w, because if $r = \min(w - a, b - w)$ and $x \in (a, b)$ then $w - r < x < w + r$.

Neighbourhoods and continuity

$f : \mathbf{R} \to \mathbf{R}$ is continuous at $w \in \mathbf{R}$ if and only if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $|w - x| < \delta$ then $|f(w) - f(x)| < \epsilon$.

Let's re-express this in terms of neighbourhoods. The set of x for which $|f(w) - f(x)| < \epsilon$ contains the open ball $|w - x| < \delta$, so is a neighbourhood of w. This is the set where $f(x) \in B(f(w), \epsilon)$, i.e. where $x \in f^*(B(f(w), \epsilon)$. So $f^*(B(f(w), \epsilon))$ is a neighbourhood of w.

If Z is a neighbourhood of $f(w)$ then there is an $\epsilon > 0$ such that $B(f(w), \epsilon) \subseteq Z$. Then $f^*(B(f(w), \epsilon)) \subseteq f^*(Z)$. So $f^*(Z)$ is a neighbourhood of w. If f is continuous at w then for every neighbourhood Z of $f(w)$ the preimage $f^*(Z)$ is a neighbourhood of w.

The converse is also true. Suppose that for every neighbourhood Z of $f(w)$ the preimage $f^*(Z)$ is a neighbourhood of w. $B(f(w), \epsilon)$ is a neighbourhood of $f(w)$. So $f^*(B(f(w), \epsilon))$ is a neighbourhood of w. There then is a $\delta > 0$ such that $B(w, \delta) \subseteq f^*(B(f(w), \epsilon))$, i.e such that if $|w - x| < \delta$ then $|f(w) - f(x)| < \epsilon$. So f is continuous at w.

Neighbourhoods and continuity, continued

So, for functions $f : \mathbf{R} \to \mathbf{R}$ we find that f is continuous at $w \in \mathbf{R}$ if and only if for every neighbourhood Z of $f(w)$ the set $f^*(Z)$ is a neighbourhood of w.

Unlike the original definition, this condition makes sense in any topological space, so we can use it to *define* continuity for functions from one topological space to another.

We write $\mathcal{N}(x)$ for the of all neighbourhoods of x in a given topological space. We can rewrite the definition as f is continuous at w if and only if for every neighbourhood $Z \in \mathcal{N}(f(w))$, $f^*(Z) \in \mathcal{N}(w)$, i.e. $Z \in f^{**}(\mathcal{N}(w))$. In other words, f is continuous at w if and only if

$$\mathcal{N}(f(w)) \subseteq f^{**}(\mathcal{N}(w)).$$

There's a similar criterion for $\lim_{x \to w} f(x) = z$, but with $\mathcal{N}(z)$ in place of $\mathcal{N}(f(w))$ and something complicated in place of $\mathcal{N}(w)$.