MAU22200 Lecture 5

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Properties of open sets in a metric space

Recall that open balls in a metric space (X, d) are defined by $B(x, r) = \{y \in X : d(x, y) < r$ for $r > 0$. Also that $S \subseteq X$ is open if and only if it contains an open ball about each of its points, i.e. for all $x \in S$ there is an $r > 0$ such that $B(x, r) \subseteq S$. We saw last time that if (X, d) is a metric space then X and $\varnothing$ are open (and also closed) subsets of X.
Two other useful facts are that finite intersections of open sets are open and arbitrary unions of open sets are open.
We'll start with the intersections. It's enough to prove that the intersection of two open sets is open. The general case follows by induction on the number of sets.

Intersections

Suppose V and W are open. If $x \in V \cap W$ then $x \in V$ and $x \in W$. $x \in V$ and V is open so there is an $s > 0$ such that $B(x,s) \subseteq V$. $x \in W$ and W is open so there is a $t > 0$ such that $B(x,t) \subseteq W$. There's no reason why t would be equal to s. Let $r = \min(s,t)$. Then $B(x,r) \subseteq B(x,s)$ and $B(x,r) \subseteq B(x,t)$. So $B(x,r) \subseteq V$ and $B(x,r) \subseteq W$. Therefore $B(x,r) \subseteq V \cap W$. For an arbitrary $x \in V \cap W$ we found an $r > 0$ such that $B(x,r) \subseteq V \cap W$. So $V \cap W$ is open.

It wouldn't be hard to prove the result for finite intersections directly, without induction. If $x \in \cap_{j=1}^{n} V_j$ and V_j is open for each j then choose s_j such that $B(x,s_j) \subseteq V_j$. Let $r = \min_{1 \le j \le n} s_j$. Then $B(x,r) \subseteq \cap_{j=1}^{n} V_j$.

It's clear where finiteness is used: the minimum of finitely many positive numbers is positive. The corresponding statement for infinite intersections of open sets is false, as the example $\{0\} = \cap_{j=1}^{\infty}(-1/j, 1/j)$ shows.

Unions

The statement about unions is conceptually easier but notationally harder.
Suppose $\mathcal{E}$ is a set of open sets. Then $\bigcup_{V \in \mathcal{E}} V$ is open.
As a notational convention I try to use lower case letters, e.g. x, y, for points, upper case letters, e.g. X, Y for sets of points and script letters, e.g. $\mathcal{E}$, $\mathcal{T}$, for sets of sets of points. This will sometimes break down. For example, the "points" in the Hamming metric example are sets.
If $x \in \bigcup_{V \in \mathcal{E}} V$ then $x \in V$ for some $V \in \mathcal{E}$. V is open so there is an $r > 0$ such that $B(x, r) \subseteq V$. But $V \subseteq \bigcup_{V \in \mathcal{E}} V$. If mixing free and bound variables makes you uneasy then write this as $V \subseteq \bigcup_{W \in \mathcal{E}} W$. In any case, $B(x, r) \subseteq \bigcup_{V \in \mathcal{E}} V$. So $\bigcup_{V \in \mathcal{E}} V$ is open.

Open sets and limits

It's possible to describe limits without referring to the metric or to balls directly.

> *If (X, d_X) and (Y, d_Y) are metric spaces, $w \in X$, $z \in Y$ and $f : X \to Y$ is a function then $\lim_{x \to w} f(x) = z$ if and only if for every open $Z \in \wp(Y)$ such that $z \in Z$ there is an open $W \in \wp(X)$ such that $w \in W$ and*
>
> $$W \setminus \{w\} \subseteq f^*(Z).$$

The version in the notes is more complicated because it applies to a function f defined on a subset U of X which needn't have w as an element.

Note that d_X and d_Y don't appear, except indirectly since open sets are defined in terms of open balls and open balls are defined in terms of metrics.

Topologies

The next step in our progressive generalisation is a bit of a leap compared to the previous ones.
A topology on a set X is a $\mathcal{T} \in \wp(\wp(X))$ such that

- $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$.
- If $V, W \in \mathcal{T}$ then $V \cap W \in \mathcal{T}$.
- If $\mathcal{E} \subseteq \mathcal{T}$ then $\bigcup_{V \in \mathcal{E}} V \in \mathcal{T}$.

The set of open sets in a metric space is a topology. In fact the definition was obtained by abstracting the properties proved previously for open sets in a metric space. We've effectively turned that theorem into a definition. Or rather into the definition of a topology plus the theorem that open sets in a metric space form a topology.

Limits in topological spaces

We can now define limits in the context of topological spaces by replacing open sets by elements of $\mathcal{T}$.

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, $w \in X$, $z \in Y$ and $f : X \to Y$ is a function then $\lim_{x \to w} f(x) = z$ if (and only if) for every $Z \in \mathcal{T}_Y$ such that $z \in Z$ there is a $W \in \mathcal{T}_X$ such that $w \in W$ and

$$W \setminus \{w\} \subseteq f^*(Z).$$

Again, the version in the notes is more complicated because it doesn't assume f is defined on all of X.

We can take theorems for limits in metric spaces and try to adapt to topological space. Ignore, for now, the question of why we'd want to. This is a more complicated operation than moving from $\mathbf{R}^n$ to normed vector spaces or from normed vector spaces to metric spaces.

Hausdorff topologies

If we try to prove the uniqueness of limits in the context of topological spaces we find that we can't quite do it. The notion of a topology captured some of the properties of open sets in a metric space, but not all of them, and not enough of them.
Another property of open sets in a metric space is that if $x \neq y$ then there are open sets V and W such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$. Various choices of V and W work. One is given in the notes.
The corresponding statement for topological spaces would be that if $x \neq y$ then there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$. That's not true for all topologies though.
$\mathcal{T} = \{\varnothing, X\}$ satisfies all the requirements to be a topology, but doesn't have the property above if $\#X > 1$. This topology is called the *trivial topology*.

Hausdorff topologies (continued)

We could add this extra condition to the definition of a topology, but

- there are useful examples, more interesting than the trivial topology, where it fails, and
- that's not the way people have been using the word for the last century.

Instead we introduce a definition. A topology $\mathcal{T}$ is called *Hausdorff* if whenever $x \neq y$ there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$. The topology of open sets in a metric space is always Hausdorff but the trivial topology is only Hausdorff if $\#X \leq 1$.
As an aside, this shows that the trivial topology does not come from a metric.

Rescuing uniqueness

If we try to prove the uniqueness of limits in the context of topological spaces we find that we can't quite do it.
In the context of Hausdorff topological spaces it works though.
You don't need both topologies to be Hausdorff though. Only $\mathcal{T}_Y$ needs to be Hausdorff.
There's another hypothesis that's needed, this time on $(X, \mathcal{T}_X)$. This one was already needed in the metric space case. It's a consequence of the awkward $0 < |x - w|$ in the historic definition of limits. That because a $\setminus\{w\}$ in our generalisation. The extra hypothesis we need is that w is a *limit point* of the domain, U, of the function f. That means that for every $W \in \mathcal{T}$ such that $w \in W$ we have $U \cap W \setminus \{w\} \neq \emptyset$.