MAU22200 Lecture 4

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Balls

Once you have a metric you can define balls and related notions. If (X, d) is a metric space, $x \in X$ and $r > 0$ then

- $\{y \in X : d(x, y) < r\}$ is the *open* ball of radius r about, i.e. centred at, x. It will be denoted $B(x, r)$.
- $\{y \in X : d(x, y) \leq r\}$ is the *closed* ball of radius r about x. It will be denoted $\bar{B}(x, r)$.
- $\{y \in X : d(x, y) = r\}$ is the sphere of radius r about x. We won't bother with a notation for it.

Note that "ball" and "sphere" are not synonyms! Also, all balls are of positive radius. It's not necessary to say this explicitly but the words "of positive radius" may sometimes be added for emphasis.

Comparison with balls in $\mathbf{R}^n$

If $X = \mathbf{R}^n$ and $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ then these agree with the usual definitions, at least if you're someone who distinguishes between balls and spheres.
Most of the properties of balls in $\mathbf{R}^n$ hold in general metric spaces. For example, if $y \in B(x, r)$ then

$$B(y, r - d(x, y)) \subseteq B(x, r).$$

The radius is positive, as required, because these are open balls.
The inclusion is a consequence of the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z).$$

If the distance from y to z is less than $r - d(x, y)$ then the distance from x to z is less than r.

Comparison with balls in $\mathbf{R}^n$ (continued)

Other properties do not carry over. Spheres can be empty, for example! Equivalently, the open ball of radius r about x is a subset of the closed ball of radius r about x, but not necessarily a *proper* subset. The discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

provides a counter-example. $B(x, 1/2) = \bar{B}(x, 1/2) = \{x\}$. $B(x, 3/2) = \bar{B}(x, 3/2) = X$. More generally, $B(x, r) = \bar{B}(x, r)$ unless $r = 1$.
Also, if $r < s$ then $B(x, r) \subseteq B(x, s)$, but it needn't be a proper subset. For the discrete metric $B(x, 1/3) = B(x, 2/3)$.

Why balls?

You obviously can define balls this way, but why would you want to? They enable you to rewrite inequalities like $d_Y(f(x), z) < \epsilon$ and $0 < d_X(x, w) < \delta$ in terms of set membership as $f(x) \in B(z, \epsilon)$ and $x \in B(w, \delta) \setminus \{w\}$. The first of these is just the definition. For the second, note that $d_X(x, w) < \delta$ if and only if $x \in B(w, \delta)$ while $0 < d_X(x, w)$ if and only if $x \neq w$.
The definitions of limits, continuity, etc. have strict inequalities, i.e. $<$, rather than weak inequalities, i.e. $\leq$, so open balls appear more frequently than closed balls.
Introducing balls is one step in replacing statements about points with statements about sets. The next step is to introduce images and preimages.

Image and preimage

If φ is a function from a set X to a set Y, U is a subset of X and V is a subset of Y then

- $\{x \in X : \varphi(x) \in V\}$ is called the preimage of V under φ. It's denoted by $\varphi^*(V)$. Some people write it as $\varphi^{-1}(V)$, but that's dangerous.
- $\{y \in Y : \exists x \in U : y = \varphi(x)\}$ is called the image of V under φ. It's denoted by $\varphi_*(U)$. Some people write it as $\varphi(U)$, but that's dangerous.

φ_* and φ^* are functions, but from what set to what set? φ_* is a function from $\wp(X)$ to $\wp(Y)$. φ^* is a function from $\wp(Y)$ to $\wp(X)$. $\wp$ denotes the power set, i.e. set of subsets.

The image and preimage have many properties, which are listed in the notes. You need to become familiar with these. I'll generally use them without comment.

Preimages and limits

The statement $d_Y(f(x), z) < \epsilon$, which we saw was equivalent to

$$f(x) \in B(z, \epsilon),$$

is also equivalent to

$$x \in f^*(B(z, \epsilon)).$$

The statement "if $0 < d_X(x, w) < \delta$ then $d_Y(f(x), z) < \epsilon$" is equivalent to "if $x \in B(w, \delta) \setminus \{w\}$ then $x \in f^*(B(z, \epsilon))$". Or just

$$B(w, \delta) \setminus \{w\} \subseteq f^*(B(z, \epsilon)).$$

The variable x has disappeared. A conditional statement about points has become and inclusion of sets.

Open and closed balls

The words "open" and "closed" in open and closed balls have an independent meaning. $U \in \wp(X)$ is called open if it contains an open ball about each of its points. In other words, for each $x \in U$ there is an $r > 0$ such that $B(x, r) \subseteq U$.
Closed does not mean what you might expect! It does not mean that it contains a closed ball about each of its points. It also doesn't mean not open. $V \in \wp(X)$ is called closed if $X \setminus V$ is open.
The terminology is confusing at first, but it is at least consistent in the sense that open balls are open and closed balls are closed. For open balls that the inclusion

$$B(y, r - d(x, y)) \subseteq B(x, r).$$

from earlier.

Intervals

Intervals in $\mathbf{R}$ are a good place to get an intuition. There are ten types of interval:

- (a, b), where $a < b$
- $[a, b]$, where $a \leq b$
- $[a, b)$, where $a < b$
- $(a, b]$, where $a < b$
- $(a, +\infty)$
- $[a, +\infty)$
- $(-\infty, b)$
- $(-\infty, b]$
- $(-\infty, \infty)$
- $\varnothing$

More intervals

- (a, b) is open. If $x \in (a, b)$ then $B(x, \min(x - a, b - x)) \subseteq (a, b)$. (a, b) is not closed. In other words $\mathbf{R} \setminus (a, b)$ is not open. $a \in \mathbf{R} \setminus (a, b)$ but there is no $r > 0$ such that $B(a, r) \subseteq \mathbf{R} \setminus (a, b)$.
- $[a, b]$ is closed. In other words $\mathbf{R} - [a, b]$ is open. If $x \in \mathbf{R} - [a, b]$ then $B(x, \min(|x - a|, |x - b|) \subseteq \mathbf{R} - [a, b]$. $[a, b]$ is not open. $a \in [a, b]$ but there is no $r > 0$ such that $B(a, r) \subseteq [a, b]$.
- $[a, b)$ is neither open nor closed. It's not open because $a \in [a, b)$ but there is no $r > 0$ such that $B(a, r) \subseteq [a, b)$. It's not closed because $b \in \mathbf{R} \setminus [a, b)$ but there is no $r > 0$ such that $B(b, r) \subseteq \mathbf{R} \setminus [a, b)$.
- Similarly, $(a, b]$ is neither open nor closed.

Still more intervals

- $(a, +\infty)$ and $(-\infty, b)$ are open, but not closed.
- $[a, +\infty)$ and $(-\infty, b]$ are closed, but not open.
- $(-\infty, +\infty)$ is open and closed! It's open because if $x \in \mathbf{R}$ then there is an $r > 0$ such that $B(x, r) \subseteq \mathbf{R}$. In fact any r will work. It's closed because if $x \in \mathbf{R} \setminus \mathbf{R}$ then there is an $r > 0$ such that $B(x, r) \subseteq \mathbf{R} \setminus \mathbf{R}$. You can ignore everything after the word "then" because the condition $x \in \mathbf{R} \setminus \mathbf{R}$ is never satisfied.
- Similarly, $\varnothing$ is both open and closed.

The last two statements can be generalised to any metric space. If (X, d) is a metric space and X and $\varnothing$ are open subsets of X, and also closed subsets of X.
There may or may not be other sets which are both open and closed.

The discrete metric

Suppose d is the discrete metric on X. Which subsets are open? Which are closed?
If $S \in \wp(X)$ then S is open. If $x \in S$ then $B(x, 1/2) = \{x\} \subseteq S$.
If $S \in \wp(X)$ then S is closed, because $X \setminus S \in \wp(X)$ and so $X \setminus S$ is open. All subsets of X are both open and closed with respect to the discrete metric.
This tells you, if you hadn't already guessed, that the open and closed sets depend on the choice of the metric. $\mathbf{R}$ with the usual metric has sets which are neither open nor closed. With the discrete metric all subsets are both open and closed.
It's not uncommon though for two different metrics to give rise to the same collection of open sets.