MAU22200 Lecture 3

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From $\mathbf{R}$ to $\mathbf{R}^n$

Generalising results from $\mathbf{R}$ to $\mathbf{R}^n$ is mostly straightforward, except:

- Some real numbers have to stay real, e.g. δ and ϵ in the definition of limits.
- We may want to consider spaces of more than one dimension, e.g. functions from $\mathbf{R}^m$ to $\mathbf{R}^n$ where m is not necessarily equal to n.
- Some structures on $\mathbf{R}$ lack good analogues in $\mathbf{R}^n$, e.g. the order structure of multiplicative structure.
- Some statements which would be equivalent in $\mathbf{R}$ aren't in $\mathbf{R}^n$. For example, the product rule for differentiation generalises to the cross product in $\mathbf{R}^3$ if you write it as $(fg)' = f'g + fg'$ but not if you write it as $(fg)' = fg' + gf'$.
- $\mathbf{R}^0$ can cause problems either in statements or proofs.

How to do it, and how not to

The way to do this is to take statements and proofs and

- replace $\mathbf{R}$ with $\mathbf{R}^m$, $\mathbf{R}^n$, etc. as appropriate,
- replace absolute values by lengths, where needed,
- write vectors and vector valued functions in bold or with arrows, if using one of those conventions, and
- check that all expressions are meaningful, e.g. that you aren't adding scalars to vectors.

This is to be done in definitions, statements of theorems and proofs. With rare exceptions it works.

What *not* to do is to define vector concepts componentwise. It's a bad idea, for example, to define $\lim_{\mathbf{x}\to\mathbf{w}} \mathbf{f}(\mathbf{x}) = \mathbf{z}$ as meaning that $\lim_{\mathbf{x}\to\mathbf{w}} f_j(\mathbf{x}) = z_j$ for $j = 1, 2, \ldots, n$. It's tempting, because you can recycle results from the real case, but it leads to trouble.

From $\mathbf{R}^n$ to normed vector spaces

Some properties of the Euclidean norm aren't really needed. Like $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$.
Others are used frequently, like

- $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$,
- $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$, and
- $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

It's useful to extract these frequently used properties and define a norm on a vector space to be a real valued function which satisfies them, i.e. $q\colon V \to \mathbf{R}$ is a norm if (and only if)

- $q(\mathbf{x}) \geq 0$ and $q(\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$,
- $q(\alpha\mathbf{x}) = |\alpha| q(\mathbf{x})$, and
- $q(\mathbf{x}+\mathbf{y}) \leq q(\mathbf{x}) + q(\mathbf{y})$.

To adapt definitions, theorems and proofs to normed vector spaces, replace the Euclidean norm with a general norm (or norms).

Examples

It's possible to find other norms on $\mathbf{R}^n$ besides the usual (Euclidean) norm.

$$q(\mathbf{x}) = \left(\sum_{j=1}^{n} \left| x_j \right|^p \right)^{1/p}$$

is a norm for any $p \in [1, \infty)$. For $p = 2$ this is the Euclidean norm.

It's also possible to define norms on infinite dimensional spaces. $C([a, b])$, the space of continuous functions from $[a, b]$ to $\mathbf{R}$, is a vector space.

$$q(f) = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p}$$

is a norm for any $p \in [1, \infty)$.

Metrics

In discussing limits, continuity, etc. we only really need lengths (or norms) of *differences* of vectors, i.e. distances between vectors. Again, we only need certain properties:

- $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$,
- $d(x, y) = d(y, x)$, and
- $d(x, z) \leq d(x, y) + d(y, z)$.

A function satisfying these properties is called a metric. I've dropped the bold face font because these properties don't reference addition or scalar multiplication, so make sense beyond vector spaces.
Given a norm q we can always get a metric d by taking $d(\mathbf{x}, \mathbf{y}) = q(\mathbf{x} - \mathbf{y})$, but not all metrics arise in this way.

Examples

Subsets of metric spaces are metric spaces, e.g. $S^2 \in \mathbf{R}^3$. Just restrict the metric from the larger space.

Other examples of metrics include the following

- The discrete metric on a set S: $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise.
- The Hamming distance on the subsets of a finite set S: $d(A, B) = \#A \triangle B$, where $\#$ represents cardinality and $\triangle$ represents symmetric difference.
- The p-adic metric on the set $\mathbf{Q}$ of rational numbers.

Successive generalisations

You can track through a definition or proof as we go through these various levels of generalisation.

$$|f(x) - z| < \epsilon$$

in the definition of the limit of a real valued function of a real variable becomes

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{z}\| < \epsilon$$

for functions form $\mathbf{R}^m$ to $\mathbf{R}^n$ and then

$$q(\mathbf{f}(\mathbf{x}) - \mathbf{z}) < \epsilon$$

for functions from a normed vector space to a normed vector space and then

$$d(f(x), z) < \epsilon$$

for functions from a metric space to a metric space.