MAU22200 Lecture 1

John Stalker

Trinity College Dublin

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Module information

- Module: MAU22200 Advanced Analysis
- Instructor: John Stalker (`stalker@maths.tcd.ie`)
- Webpage: `https://www.maths.tcd.ie/~stalker/22200/`
- I will use try to post everything both to the webpage and to Blackboard.
- That applies in particular to lectures and lecture notes.
- Marking will be 80% exam, 20% continuous assessment.
- There will also be practice problems, unmarked, but with solutions posted.
- There are 3 lectures per week, posted online, and a weekly tutorial?
- There is a discussion board on Blackboard.

Module content

- General topology
  - Introduction
  - Cardinality of sets
  - Topological spaces
  - Metric spaces
  - Normed spaces
  - Filters and nets
- Measure and integration

A more detailed outline is on the module webpage, but most of the words won't mean much to you for now.

What is general topology?

General topology is the systematic generalisation of certain arguments from elementary real and complex analysis, particularly those related to limits and continuity.
Why do we generalise (in general)?

- to avoid repeating similar arguments
- to include new examples
- to eliminate distractions
- to limit our options when proving theorems
- because that's just what mathematicians do

Example (limits)

In real analysis limits are defined as follows

> $\lim_{x\to w} f(x) = z$ *if and only if for all* $\epsilon > 0$ *there is a* $\delta > 0$ *such that if* $0 < |x - w| < \delta$ *then* $|f(x) - z| < \epsilon$.

Limits have a number of interesting properties. For example,

- If $f = g + h$ then $\lim_{x\to w} f(x) = \lim_{x\to w} g(x) + \lim_{x\to w} h(x)$.
- If $f \leq g$ then $\lim_{x\to w} f(x) \leq \lim_{x\to w} g(x)$.

There are many more types of limits though.

- limits at infinity, i.e. $\lim_{x\to+\infty} f(x) = z$ if and only if for all $\epsilon > 0$ there is an M such that if $x \geq M$ then $|f(x) - z| < \epsilon$
- limits of sequences, i.e. $\lim_{n\to\infty} \alpha_n = z$ if and only if for all $\epsilon > 0$ there is an N such that if $n \geq N$ then $|\alpha_n - z| < \epsilon$.
- limits from above or below, i.e. $\lim_{x\searrow w} f(x) = z$ if and only if ... This one's also written $\lim_{x\to w^+} f(x)$.

Common features, and differences

The definitions of all these types of limits have a certain family resemblance. They also have common properties. For example, if $\alpha = \beta + \gamma$ then $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n + \lim_{n\to\infty} \gamma_n$. Also, if $\alpha \leq \beta$ then $\lim_{n\to\infty} \alpha_n \leq \lim_{n\to\infty} \beta_n$. There are similar theorems for limits at infinity or for limits from above or below.
Not every generalisation preserves all the properties though. We can consider limits of vector valued functions, i.e. functions to $\mathbf{R}^n$.

> $\lim_{x\to w} \mathbf{f}(x) = \mathbf{z}$ *if and only if for all* $\epsilon > 0$ *there is a* $\delta > 0$ *such that if* $0 < |x - w| < \delta$ *then* $\|\mathbf{f}(x) - \mathbf{z}\| < \epsilon$.

The limit of the sum is still the sum of the limits, but the other property has no nice analogue. $\mathbf{f} \leq \mathbf{g}$ isn't even a meaningful hypothesis.

Limits and sums and integrals

Infinite sums can be thought of as limits of sequences of partial sums. They inherit the properties of limits of sequences in general. If $\alpha = \beta + \gamma$ then $\sum_{j=1}^{\infty} \alpha_j = \sum_{j=1}^{\infty} \beta_j + \sum_{j=1}^{\infty} \gamma_j$, for example. Similarly, if $\alpha \le \beta$ then $\sum_{j=1}^{\infty} \alpha_j \le \sum_{j=1}^{\infty} \beta_j$.
It's also true that if $f = g + h$ then

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx + \int_a^b h(x)\,dx$$

and if $f \le g$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Riemann integrals are "limits" of Riemann sums, but in what sense? Can one make this precise and use it to prove the properties above?

Levels of generalisation

We can define limits in various levels of generality. More general means more abstract, but also covering more cases. This can get very abstract. The first chapter of the notes goes through the following levels.

- Real valued functions of a real variable (defined everywhere)
- Vector ($\mathbf{R}^n$) valued functions of a vector ($\mathbf{R}^m$) variable (defined everywhere)
- Functions from a normed vector space to a normed vector space (defined everywhere)
- Functions from a metric space to a metric space (defined on some, not necessarily proper, subset)
- Functions from a topological space to a topological space (defined on some subset)
- Functions from a filtered space to a filtered space (defined on some subset)

This semester and next semester

These generalisations are not just important for limits. They have many other uses. This semester is mostly concerned with the intermediate levels of abstraction: normed vector spaces, metric spaces, and topological spaces. Those aren't quite enough to realise integrals as limits. For that you need filters.

Integration is the main topic of next semester. You've seen the Riemann integral in one dimension. We want to be able to integrate functions defined on subsets of $\mathbf{R}^2$, $\mathbf{R}^3$ or $\mathbf{R}^m$. There is a theory of Riemann integration for such functions. Riemann integration is fine for continuous functions or nearly continuous functions, but it has some drawbacks. The main purpose of the next semester is to introduce an extension of the Riemann integral: the Lebesgue integral.