Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

SF Maths **2021-2022**
SF TJH

MAU22200 Advanced Analysis

Wednesday 4 May RDS Main Hall 14:00 — 17:00

Prof. John Stalker

Instructions to Candidates:

Answer two questions from Part A and two questions from Part B. In questions with multiple parts you are allowed to use the results of earlier parts in later parts, whether or not you did those parts.

Materials Permitted for this Examination:

Formulae and Tables are available from the invigilators, if required.

Calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

Part A

1. (20 points) Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Suppose that A is a dense subset of X, i.e. that the closure of A is X. Suppose that $f\colon A \to Y$ is a function. We say that $g\colon X \to Y$ is an extension of f if $g(x) = f(x)$ for all $x \in A$.

   (a) (10 points) Suppose that f is continuous. Prove that f has at most one continuous extension.

   Hint: If g_1 and g_2 are extensions then try showing that

   $$\{x \in X\colon g_1(x) = g_2(x)\}$$

   is a closed set in X which contains A.

   (b) (10 points) Prove that if (Y, d_Y) is complete and f is uniformly continuous then f has at least one continuous extension.

2. (20 points)

(a) (4 points) State, but do not prove, the Heine-Borel Theorem.

(b) (8 points) Give an example of a metric space (X, d) and a set $K \subseteq X$ such that K is closed and bounded, but not compact.

(c) (8 points) The Extreme Value Theorem says that any continuous real valued function on a non-empty compact set has a minimum and a maximum on that set. Prove the following partial converse for subsets of $\mathbf{R}^n$:

> Suppose $K \subseteq \mathbf{R}^n$ is such that every continuous function $f \colon K \to \mathbf{R}$ has a minimum and maximum on K. Then K is compact.

Hint: Consider the functions $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{y} \in \mathbf{R}^n$.

3. (20 points)

   (a) (12 points) Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, $\mathcal{C}$ is an open cover of X and $f\colon X \to Y$ is a function. Prove that f is continuous if and only if its restriction to each $U \in \mathcal{C}$ is continuous.

   (b) (8 points) Show, by means of an example, that the statement above can fail if we merely assume that $\mathcal{C}$ is a cover rather than an open cover.

Part B

4. (20 points) Suppose $(X, \mathcal{B}, \mu)$ is a measure space. For any measurable function $f\colon X \to [-\infty, +\infty]$ we define the essential supremum and essential infimum of f by

$$\operatorname{ess\,sup} f = \inf\{y \in [-\infty, +\infty]\colon \mu(f^*((y, +\infty])) = 0\}$$

and

$$\operatorname{ess\,inf} f = \sup\{y \in [-\infty, +\infty]\colon \mu(f^*([-\infty, y))) = 0\}.$$

where f^* is the preimage

$$f^*(E) = \{x \in X\colon f(x) \in E\}.$$

f is said to be essentially bounded if

$$\operatorname{ess\,inf} f > -\infty$$

and

$$\operatorname{ess\,sup} f < +\infty.$$

(a) (12 points) Prove the following statements.

i. (3 points) If f and g are measurable functions such that $f(x) = -g(x)$ for all $x \in X$ then $\operatorname{ess\,inf} f = -\operatorname{ess\,sup} g$.

ii. (3 points) If f and g are measurable functions such that $f(x) \le g(x)$ for all $x \in X$ then $\operatorname{ess\,inf} f \le \operatorname{ess\,inf} g$ and $\operatorname{ess\,sup} f \le \operatorname{ess\,sup} g$

iii. (3 points) If $\mu(X) > 0$ and f is a measurable function then $\operatorname{ess\,inf} f \le \operatorname{ess\,sup} f$.

iv. (3 points) If f, g and h are measurable functions such that $f = g + h$ then $\operatorname{ess\,sup} f \le \operatorname{ess\,sup} g + \operatorname{ess\,sup} h$.

(b) (8 points) Find an example of a measurable function which is essentially bounded but not bounded.

Note: You're allowed to choose any measure space you want but you might find it convenient to choose $\mathbf{R}$ with Lebesgue measure.

5. (20 points) Suppose $(X, \mathcal{B}, \mu)$ is a measure space and E_0, E_1, ... is a sequence of elements of $\mathcal{B}$. Let F be the set of $x \in X$ such that $x \in E_n$ for all but finitely many $n \in \mathbf{N}$. Let G be the set of all $x \in X$ such that $x \in E_n$ for infinitely many $n \in \mathbf{N}$

   (a) (2 points) Prove that $F = \bigcup_{m \geq 0} \bigcap_{n \geq m} E_n$.

   (b) (2 points) Prove that $G = \bigcap_{m \geq 0} \bigcup_{n \geq m} E_n$.

   (c) (2 points) Prove that

   $$\chi_F = \sup_{m \geq 0} \inf_{n \geq m} \chi_{E_n}$$

   where χ denotes characteristic functions, i.e.

   $$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

   (d) (2 points) Prove that

   $$\chi_G = \inf_{m \geq 0} \sup_{n \geq m} \chi_{E_n}$$

   (e) (4 points) Prove that $\lim_{n \to \infty} \chi_{E_n}(x)$ is equal to 1 if $x \in F$, is equal to 0 if $x \notin G$, and does not exist if $x \in G \setminus F$.

   (f) (8 points) Suppose that there is an $H \in \mathcal{B}$ such that $E_j \subseteq H$ for all j and $\mu(H) < +\infty$. Suppose also that $\mu(G \setminus F) = 0$. Prove that

   $$\mu(F) = \lim_{j \to \infty} \mu(E_j) = \mu(G).$$

   Hint: Apply the Dominated Convergence Theorem.

6. (20 points) For each Lebesgue measurable subset E of $\mathbf{R}$ let $\varphi(E)$ be the set

$$\varphi(E) = \left\{ x \in \mathbf{R}\colon \lim_{h \searrow 0} \frac{1}{2h} m(E \cap [x-h, x+h]) = 1 \right\},$$

where m is Lebesgue measure. This $\varphi(E)$ is a Lebesgue measurable subset, a fact which you may use without proof in the following.

(a) (8 points) Show if $E = (-1,0) \cup (0,1) \cup \{2\}$ then there is a point which is in E but not in $\varphi(E)$ and a point which is in $\varphi(E)$ but not in E.

(b) (12 points) Show that

$$m(E \triangle \varphi(E)) = 0,$$

where $\triangle$ denotes the symmetric difference

$$E \triangle \varphi(E) = (E \setminus \varphi(E)) \cup (\varphi(E) \setminus E).$$

Hint: Apply the Lebesgue Differentiation Theorem, i.e. the version of the First Fundamental Theorem of Calculus in terms of averages rather than derivatives, to the characteristic function of E.