Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science
School of Mathematics

SF Maths **Semesters 1 & 2 2020-2021**
SF JH
JS TSM

MAU22200 Advanced Analysis

Thursday 13 May 2021 Take home exam 12:00 — 18:00

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Instructions that apply to all take-home exams:

1. This is an open-book exam. You are allowed to use your class notes, textbooks and any material which is directly linked from the module's Blackboard page or from the module's webpage, if it has one. You may not use any other resources except where your examiner has specifically indicated in the "Additional instructions" section below. Similarly, you may only use software if its use is specifically permitted in that section. You are not allowed to collaborate, seek help from others, or provide help to others.

2. If you have any questions about the content of this exam, you may seek clarification from the lecturer using the e-mail address provided. You are not allowed to discuss this exam with others. You are not allowed to send exam questions or parts of exam questions to anyone or post them anywhere.

3. Unless otherwise indicated by the lecturer in the "Additional instructions" section, solutions must be submitted through Blackboard in the appropriate section of the module webpage by the deadline listed above. You must submit a single pdf file for each exam separately and sign the following declaration in each case. It is your responsibility to check that your submission has uploaded correctly in the correct section.

Additional instructions for this particular exam:

Credit will be given for the best two questions from Part A (Questions 1, 2, and 3) and the best two from Part B (Questions 4, 5, and 6).

Plagiarism declaration: I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar which are available through https://www.tcd.ie/calendar.

Signature: ______________________________

Part A (Semester 1)

1. (a) (12 points) Show that

$$d(x,y) := |\log(x) - \log(y)|$$

defines a metric on $X := \{x \in \mathbb{R} : x > 0\}$.

(b) (13 points) Let X, d be as in part (a). Is (X,d) a complete metric space?

Justify your answers.

2. (a) (12 points) Consider the topological space $X = \mathbb{Z} \times \mathbb{R}$ with the product topology of the standard topologies on $\mathbb{Z}$ and $\mathbb{R}$. Find the interior, closure and boundary of the subset

$$A = \{0, 1\} \times [0, +\infty) \subset X.$$

(b) (13 points) Prove or disprove: For every continuous map $f\colon X \to Y$ between topological spaces X and Y, if $K \subset Y$ is compact then $f^{-1}(K)$ is compact.

Justify your answers.

3. (a) (12 points) Prove or disprove: for all normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the subset

$$\{(x, y) \in X \times Y : \|x\| \leq \|y\|\}$$

is closed in $X \times Y$ with respect to the product 2-norm.

(b) (13 points) Find the operator norm of

$$A\colon \ell^1 \to (\mathbb{R}^2, \|\cdot\|_\infty), \quad Ax := (x_1, 2x_2 + 3x_3), \quad x = (x_n)_{n\geq 1} \in \ell^1.$$

Justify your answers.

Part B (Semester 2)

4. If $q \in (a,b)$ and $f\colon [a,b] \to \mathbf{C}$ is Riemann integrable on $[a,p]$ for all $p \in (a,q)$ and is Riemann integrable on $[r,b]$ for all $r \in (q,b)$ then the Cauchy principal value integral of f over $[a,b]$ is defined by

$$\mathrm{PV}\int_a^b f(x)\,dx = \lim_{s\to 0^+}\left(\int_a^{q-s} f(x)\,dx + \int_{q+s}^b f(x)\,dx\right),$$

provided the limit exists.

(a) (12 points) Let $f\colon [-1,1] \to \mathbf{C}$ be defined by

$$f(x) = \begin{cases} (-x)^c & \text{if } x \in [-1,0) \\ 0 & \text{if } x = 0 \\ x^d & \text{if } x \in (0,1] \end{cases}$$

for some constants c and d. For which values of c and d

i. is f Riemann integrable?

ii. is f measurable?

iii. is f absolutely integrable?

iv. does the Cauchy principal value integral of f over $[-1,1]$ exist?

Note: Correct answers are sufficient, with no justification required. Incorrect answers may receive partial credit if a justification is given.

(b) (13 points) Prove that if f is absolutely integrable and the Cauchy principal value integral exists then the Cauchy principal value and the Lebesgue integral agree. *Note:* You need to show this for all $f\colon [a,b] \to \mathbf{C}$, not just the examples from the previous part.

5. (a) (5 points) Suppose that $f_n \colon \mathbf{R}^d \to \mathbf{C}$ are absolutely integrable functions. Prove that

$$\left|\sum_{n=1}^{N} f_n(x)\right| \leq \sum_{n=1}^{\infty} |f_n(x)|$$

for all N.

(b) (10 points) With f_n as above, prove that if

$$\sum_{n=1}^{\infty} \int_{\mathbf{R}^d} |f_n(x)|\, dx < \infty.$$

then

$$\int_{\mathbf{R}^d} \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_{\mathbf{R}^d} f_n(x)\, dx.$$

(c) (10 points) Show, by means of an example, that the equation above can fail if the hypothesis

$$\sum_{n=1}^{\infty} \int_{\mathbf{R}^d} |f_n(x)|\, dx < \infty.$$

is removed.

6. (a) (8 points) Prove that the following two conditions on a function $f\colon \mathbf{R}^d \to \mathbf{C}$ are equivalent:

   i. f is a simple function.

   ii. f is measurable and there is a non-zero polynomial p such that $p(f(x)) = 0$ everywhere.

   (b) (9 points) Show, by means of examples, that these conditions are no longer equivalent if any of the following changes are made:

   i. Dropping the assumption that f is measurable.

   ii. Dropping the assumption that p is a polynomial.

   iii. Replacing the word "everywhere" with the words "almost everywhere".

   (c) (8 points) Suppose that $f\colon \mathbf{R}^d \to \mathbf{C}$ is measurable and there is a non-zero polynomial p such that $p(f(x)) = 0$ almost everywhere. Define $g\colon \mathbf{R}^d \to \mathbf{C}$ by

$$g(x) = \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy$$

   if the limit on the right hand side exists and $g(x) = f(x)$ if it does not. Prove that $p(g(x)) = 0$ almost everywhere.