Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science
School of Mathematics

SF Maths **Semester 1 2020-2021**
SF JH
JS TSM

MAU22200 Advanced Analysis

Tuesday 13 May 2021 Take home exam 12:00 — 18:00

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Instructions that apply to all take-home exams:

1. This is an open-book exam. You are allowed to use your class notes, textbooks and any material which is directly linked from the module's Blackboard page or from the module's webpage, if it has one. You may not use any other resources except where your examiner has specifically indicated in the "Additional instructions" section below. Similarly, you may only use software if its use is specifically permitted in that section. You are not allowed to collaborate, seek help from others, or provide help to others.

2. If you have any questions about the content of this exam, you may seek clarification from the lecturer using the e-mail address provided. You are not allowed to discuss this exam with others. You are not allowed to send exam questions or parts of exam questions to anyone or post them anywhere.

3. Unless otherwise indicated by the lecturer in the "Additional instructions" section, solutions must be submitted through Blackboard in the appropriate section of the module webpage by the deadline listed above. You must submit a single pdf file for each exam separately and sign the following declaration in each case. It is your responsibility to check that your submission has uploaded correctly in the correct section.

Additional instructions for this particular exam:

Credit will be given for the best two questions from Part A (Questions 1, 2, and 3) and the best two from Part B (Questions 4, 5, and 6).

Plagiarism declaration: I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar which are available through https://www.tcd.ie/calendar.

Signature: ____

Part A (Semester 1)

1.

2.

3.

Part B (Semester 2)

4. (a) (9 points) Suppose E and F are Lebesgue measurable subsets of $\mathbf{R}^d$ with positive measure. Let G be the set of points of the form $x + y$ where $x \in E$ and $y \in F$. Prove that G has positive Lebesgue outer measure.

(b) (8 points) Give an example of Lebesgue measurable subsets E and F or $\mathbf{R}^d$ such that E has positive measure, F has measure zero and the set G of points of the form $x + y$ where $x \in E$ and $y \in F$ has measure zero.

Note: Although this can be done for every positive d you don't have to do this for every such d, just for some choice of d.

(c) (8 points) Give an example of Lebesgue measurable subsets E and F of $\mathbf{R}^d$ with measure zero such that the set G of points of the form $x + y$ where $x \in E$ and $y \in F$ has positive Lebesgue outer measure.

Note: Although this can be done for every positive d you don't have to do this for every such d, just for some choice of d.

5. (a) (10 points) Suppose $E \subset \mathbf{R}^2$ is Lebesgue measurable and has positive measure. Let F be the set of $x \in \mathbf{R}$ such that the set $\{y \in \mathbf{R} \colon (x, y) \in E\}$ is Lebesgue measurable and has positive measure. Prove that F is Lebesgue measurable and has positive measure.

   (b) (15 points) Suppose $E \subset \mathbf{R}^2$ is Lebesgue measurable and has measure zero. Let F be the set of $x \in \mathbf{R}$ such that the set $\{y \in \mathbf{R} \colon (x, y) \in E\}$ is Lebesgue measurable and has positive measure. Prove that F is Lebesgue measurable and has measure zero.

6. (a) (10 points) One of the following exists and the other doesn't:

   i. A sequence of non-negative measurable functions f_n which converges pointwise to a measurable function g such that

$$\lim_{n\to\infty} \int_{\mathbf{R}} f_n(x)\,dx = 0, \qquad \int_{\mathbf{R}} g(x)\,dx = +\infty.$$

   ii. A sequence of non-negative measurable functions f_n which converges pointwise to a measurable function g such that

$$\lim_{n\to\infty} \int_{\mathbf{R}} f_n(x)\,dx = +\infty, \qquad \int_{\mathbf{R}} g(x)\,dx = 0.$$

   Which one exists? Give an example of such f_n and g. How do you know the other doesn't exist?

   (b) (5 points) Suppose E_1, E_2, etc. are Lebesgue measurable subsets of $\mathbf{R}^d$ and

$$\sum_{n=1}^{\infty} m(E_n) < \infty.$$

   Prove that for almost all $x \in \mathbf{R}^d$ there are only finitely many n such that $x \in E_n$.

   (c) (10 points) Suppose E_1, E_2, etc. are Lebesgue measurable subsets of $\mathbf{R}^d$ and

$$\lim_{n\to\infty} m(E_n) = 0.$$

   Prove that for almost all $x \in \mathbf{R}^d$ there are infinitely many n such that $x \notin E_n$.