

MAU 22200 Week 9 Lecture 2

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1 April 2021

Miscellaneous announcements

Friday and Monday are bank holidays, so there's no Friday Q&A or Monday lecture. I'll send next week problem assignments around on Tuesday. Read the rest of Section 1.6 for next week.

The Fundamental Theorem of Calculus

The FTC is really a pair of theorems, and each of the pair has multiple versions. Here's a deliberately vague statement of each:

1. If $F(x) = \int_{[a,x]} f$ then $F' = f$.
2. If $F' = f$ then $\int_{[a,b]} f = F(b) - F(a)$.

The usual formulation for Riemann integrals is Theorems 1.6.9 and 1.6.7.

Theorem 1.6.7 is straightforward; the hypotheses are exactly those needed to make sense of the equations $F' = f$ and $\int_{[a,b]} f = F(b) - F(a)$. We assume that F is differentiable and f is Riemann integrable.

That's *not* true of Theorem 1.6.9. We need more than Riemann integrability to make this work. We assume f is continuous, which is a stronger assumption than Riemann integrability. Counterexamples show that the theorem is false if we only assume integrability.

Counterexample 1

Define $f(x) = 0$ if x is irrational and $f(x) = 1/q$ if x is rational and q is the smallest positive integer such that qx is an integer. f is Riemann integrable on $[0, 1]$. For $n > 0$ let $0 = x_1, \dots, x_m = 1$ be the points in $[0, 1]$ where $f(x) > \frac{1}{2n}$, taken in increasing order. Consider the partition into subintervals $I_1 = [x_1, x_1 + \frac{1}{4mn})$, $I_2 = [x_1 + \frac{1}{4mn}, x_2 - \frac{1}{4mn})$, $I_3 = [x_2 - \frac{1}{4mn}, x_2 + \frac{1}{4mn})$, $I_4 = [x_2 + \frac{1}{4mn}, x_3 - \frac{1}{4mn})$, $I_5 = [x_3 - \frac{1}{4mn}, x_3 + \frac{1}{4mn})$, \dots , $I_{2m-1} = [x_{m-1} + \frac{1}{4mn}, x_m - \frac{1}{4mn})$, $I_{2m} = [x_m - \frac{1}{4mn}, x_m]$. On the odd numbered subintervals $0 \leq f(x) \leq 1$. There are m such subintervals and they are of length at most $\frac{1}{2mn}$, so the sum of their lengths is at most $\frac{1}{2n}$. On the even numbered intervals $0 \leq f(x) \leq \frac{1}{2n}$. There are $m - 1$ such intervals and the sum of their lengths is at most 1 . The corresponding Riemann sum is between 0 and $\frac{1}{n}$. So f is Riemann integrable and has integral 0 . The same is true on subintervals. If $F(x) = \int_0^x f(t) dt$ then $F(x) = 0$ for all x and $F'(x) = 0$, so $F'(x) \neq f(x)$ for $x \in \mathbf{Q}$.

Counterexample 2

Let C be the middle thirds Cantor set from Exercise 1.2.9. Let K_n be the (disjoint) union of 2^n intervals $J_{n,1}, J_{n,2}, \dots, J_{n,2^n}$ of length $\frac{1}{3^n}$ with $J_{n,k}$ centred at the point $\frac{1}{2 \cdot 3^n} + 2 \sum_{i=1}^n \frac{\alpha_i}{3^i}$, where $\alpha_1 \alpha_2 \cdots \alpha_n$ is the number $j - 1$ written in binary, i.e. $j - 1 = \sum_{i=1}^n \alpha_i 2^{n-i}$. Then $[0, 1] = K_0 \supset K_1 \supset K_2 \supset \cdots$ and C is defined as $K_0 \cap K_1 \cap K_2 \cap \cdots$.

Let $f = 1_C$, the indicator function of the Cantor set. Partition $[0, 1]$ into $2^{n+1} - 1$ disjoint intervals $I_1, I_2, \dots, I_{2^{n+1}-1}$ as follows. Odd numbered intervals are defined by saying I_{2k-1} is the closed interval $J_{n,k}$. Even numbered intervals are defined by saying I_{2k} is the open interval between $J_{n,k}$ and $J_{n,k+1}$. There are 2^n odd numbered intervals, each of length $\frac{1}{3^n}$, where $0 \leq f(x) \leq 1$. There are $2^n - 1$ even numbered intervals where $f(x) = 0$. The Riemann sum is between 0 and $(\frac{2}{3})^n$. The integral is 0, so again $F(x) = \int_0^x f(t) dt = 0$. So $F'(x) \neq f(x)$ for $x \in C$.

The Lebesgue Case (First FTC)

In both counterexamples we have $F'(x) = f(x)$ almost everywhere. Is this still true if f is just Lebesgue (absolutely) integrable rather than Riemann integrable?

Yes! (Theorem 1.6.11). Also, in addition to being almost everywhere differentiable, F is continuous. Not every function which is differentiable almost everywhere is continuous.

(Exercise 1.6.1)

So we have two versions of the First Fundamental Theorem of Calculus.

1. Strong hypotheses and strong conclusion (Theorem 1.6.9):
If f is continuous then $F' = f$ everywhere.
2. Weak hypotheses and weak conclusion (Theorem 1.6.11):
If f is absolutely integrable then $F' = f$ almost everywhere.

The Lebesgue case (Second FTC)

Is it true that if $F' = f$ with f absolutely integrable in the Lebesgue sense then $\int_{[a,b]} f = F(b) - F(a)$?

Yes! (Proposition 1.6.41) This requires F to be differentiable everywhere though. It can fail if F is just differentiable almost everywhere.

A counterexample is the Cantor function. Define F_n on the intervals of the partition $I_1, I_2, \dots, I_{2^{n+1}-1}$ of $[0, 1]$ as follows. F_n is equal to $\frac{k-1}{2^n}$ at the left endpoint of I_{2k-1} , equal to $\frac{k}{2^n}$ at its right endpoint, and linear in between. F_n is constant equal to $\frac{k}{2^n}$ on I_{2k} . $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. The Cantor function is defined as $F(x) = \lim_{n \rightarrow \infty} F_n(x)$. The convergence is uniform, so F is continuous. F is differentiable on $\mathbf{R} \setminus C$ and $F'(x) = 0$ there. But $\int_{[0,1]} f = 0$ while $F(1) - F(0) = 1$.

The Lebesgue case (Second FTC), continued

Proposition 1.6.41 is nearly useless. How often do you see a function which is differentiable *everywhere* but not continuously differentiable? There's one in Exercise 1.6.2, but it's a bit artificial.

We'd like a version of the Second FTC which allows F to be differentiable almost everywhere, but we need some additional hypothesis to exclude the Cantor function. F is said to be absolutely continuous (Definition 1.6.39) if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every finite collection $(a_1, b_1), \dots, (a_m, b_m)$ of disjoint intervals with $\sum_{j=1}^m (b_j - a_j) < \delta$ one has $\sum_{j=1}^m |F(b_j) - F(a_j)| < \epsilon$. If F is absolutely continuous then it is continuous everywhere and differentiable almost everywhere. (Exercise 1.6.49). If F is absolutely continuous and $F' = f$ then $\int_{[a,b]} f = F(b) - F(a)$ (Theorem 1.6.40).

The Lebesgue case (Second FTC), continued

We have three versions of the Second FTC:

- ▶ Strong differentiability hypothesis and strong condition on the derivative (Theorem 1.6.7): F is differentiable everywhere and $f = F'$ is Riemann integrable.
- ▶ Strong differentiability hypothesis and weak condition on the derivative (Theorem 1.6.41): F is differentiable everywhere and $f = F'$ is (Lebesgue) absolutely integrable.
- ▶ Weaker differentiability hypothesis and an additional (somewhat weak) hypothesis on F (Theorem 1.6.40): F is differentiable *almost* everywhere and F is absolutely continuous.

In each case the conclusion is $\int_{[a,b]} f = F(b) - F(a)$.

Lipschitz functions

Absolute continuity is a bit hard to understand and tricky to prove. If F satisfies the Lipschitz condition, i.e. there is a C such that $|F(x) - F(y)| \leq C|x - y|$, then F is absolutely continuous. (Exercise 1.6.49). Together with Theorem 1.6.40 this implies that $\int_{[a,b]} f = F(b) - F(a)$ when F is Lipschitz and $F' = f$. That's easier to prove than Theorem 1.6.40 though (Exercise 1.6.45), and covers most cases of interest.