

MAU 22200 Week 8 Lecture 2

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25 March 2021

What gets used outside this module?

There's a lot in the text. What will you need later?

- ▶ Criteria for Lebesgue measurability of sets or functions.
- ▶ Some basic properties of measurable sets, integrals, etc., e.g. Exercise 1.4.35 and Theorem 1.4.37.
- ▶ Statements of some major (usually named) theorems:
 - ▶ You've seen Egorov (1.3.26) and Lusin (1.3.28) already.
 - ▶ There are several in Subsection 1.4.5: the monotone convergence theorem (1.4.43), Tonelli (1.4.44), Borel-Cantelli (Ex. 1.4.43), Fatou (1.4.46), the dominated convergence theorem (1.4.48).
 - ▶ There are more in later sections: the Lebesgue differentiation theorem (1.6.11, 1.6.12, 1.6.19), the Hardy-Littlewood maximal inequality (1.6.16, Ex. 1.6.11, 1.6.20), the Vitali covering lemma (1.6.22), The fundamental theorem of calculus (1.6.7, 1.6.9, 1.6.40, 1.6.41), Carathéodory extension theorem (1.7.3), Fubini-Tonelli (1.7.18, 1.7.21, 1.7.23),

What gets used outside this module? (continued)

- ▶ The density of compactly supported continuous functions in L^1 (1.3.20). This can be improved to compactly supported smooth functions in L^p .
- ▶ Some, but not all, definitions. You'll need the definitions of the various types of convergence. You generally won't need the definitions of Lebesgue measure, measurability of functions, the various types of integral, etc. You use the various equivalent conditions instead of the definitions.
- ▶ A stock of examples and counterexamples, e.g. the Cantor set, the indicator function of the rationals, the “escape” examples (Examples 1.4.39, 1.4.40, and 1.4.41). These are helpful for figuring out whether something you hope is true really is true, and often for identifying which hypotheses you need to make it true.

I can't 100% guarantee that this list is complete. Some things on the list are more important than others.

The Lebesgue Dominated Convergence Theorem

There are basic results which get used constantly, often tacitly, like the linearity of the integral $\int (af + bg) = a \int f + b \int g$. Other than those basic results, the two which get used most often are the Lebesgue Dominated Convergence Theorem (1.4.48) and Fubini-Tonelli (1.7.23).

Theorem (Lebesgue Dominated Convergence) Let (X, \mathcal{B}, μ) be a measure space and let $f_1, f_2, \dots : X \rightarrow \mathbf{C}$ be a sequence of measurable functions which converges μ -almost everywhere to a measurable limit $f : X \rightarrow \mathbf{C}$. Suppose that there is an unsigned absolutely integrable function $G : X \rightarrow [0, +\infty]$ such that $|f_n|$ is bounded pointwise μ -almost everywhere by G for each n . Then

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

The assumption that f is measurable is redundant (Exercise 1.4.29(vi)).

Other types of limits

The theorem refers to limits of sequences, but it can be used for other types of limits as well, using a bit of elementary topology.

Theorem Suppose (X, \mathcal{B}, μ) is a measure space and Y is a metric space. Suppose that $F: X \times Y \rightarrow \mathbf{C}$ is such that $F(x, y)$ is measurable as a function of x for each $y \in Y$ and that $F(x, y)$ is continuous at z as a function of y for μ -almost all $x \in X$.

Suppose further that there is an unsigned absolutely integrable function $G: X \rightarrow [0, +\infty]$ such that $|F(x, y)| \leq G(x)$ μ -almost everywhere for all $y \in Y$. Then

$$\lim_{y \rightarrow z} \int_{x \in X} F(x, y) d\mu = \int_{x \in X} \lim_{y \rightarrow z} F(x, y) d\mu = \int_{x \in X} F(x, z) d\mu.$$

The same can be done with the other main convergence theorems: the Monotone Convergence Theorem and Fatou's Lemma.

Proof of the Theorem

Otherwise there is an $\epsilon > 0$ such that for all $\delta > 0$ there is $y \in Y$ with $d_Y(y, z) < \delta$, $\left| \int_{X \in X} F(x, y) d\mu - \int_{X \in X} F(x, z) d\mu \right| \geq \epsilon$. For $\delta = 1/n$ choose such a y and call it y_n . Let f_n and f be defined by $f_n(x) = F(x, y_n)$ and $f(x) = F(x, z)$. Then $|f_n(x)| \leq G(x)$ almost everywhere for all n . Also, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ almost everywhere. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

But

$$\left| \int_X f_n(x) d\mu - \int_X f(x) d\mu \right| \geq \epsilon$$

for all n . The last two statements are inconsistent, so the theorem is proved by contradiction.

An Example (Poisson)

The Poisson Formula

$$u(x, y) = \frac{1}{\pi} \int_{z \in \mathbb{R}} \frac{y f(z)}{(x - z)^2 + y^2}$$

plays an important role in Partial Differential Equations. If f is bounded and continuous then the integrand is absolutely integrable for all (x, y) . The integrand is continuous in (x, y) everywhere except at $(z, 0)$. If $y = 0$ then the integrand is 0 except at x , so is 0 almost everywhere, and so $u(x, 0) = 0$. If there is an absolutely integrable g such that $\left| \frac{y f(z)}{(x - z)^2 + y^2} \right| \leq g(z)$ for all (x, y) then we can conclude from Lebesgue Dominated Convergence that $\lim_{y \rightarrow 0^+} u(x, y) = 0$. If $f = 1$ you can evaluate the integral explicitly:

$$u(x, y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases}$$

Poisson, continued

In this case $\lim_{y \rightarrow 0^+} u(x, y) = 1 \neq 0$, so there isn't an absolutely integrable g such that $\left| \frac{yf(z)}{(x-z)^2+y^2} \right| \leq g(z)$.

You can still use the Lebesgue Dominated Convergence Theorem to evaluate the limit but some preliminary work is needed. In

$$u(x, y) = \frac{1}{\pi} \int_{z \in \mathbf{R}} \frac{y f(z)}{(x-z)^2 + y^2}$$

make the change of variable $s = \frac{z-x}{y}$ for $y > 0$. Then

$$u(x, y) = \frac{1}{\pi} \int_{s \in \mathbf{R}} \frac{f(x+sy)}{1+s^2}.$$

$g(s) = \frac{\sup |f|}{1+s^2}$ is absolutely integrable and $\left| \frac{f(x+sy)}{1+s^2} \right| \leq g(s)$ for all $s \in \mathbf{R}$. Lebesgue Dominated Convergence gives

$$\lim_{y \rightarrow 0^+} u(x, y) = \frac{1}{\pi} \int_{s \in \mathbf{R}} \lim_{y \rightarrow 0^+} \frac{f(x+sy)}{1+s^2} = \frac{1}{\pi} \int_{s \in \mathbf{R}} \frac{f(x)}{1+s^2} = f(x).$$

Poisson, loose ends

There are two steps I skipped over.

- ▶ How do I justify the linear change of variable $s = \frac{z-x}{y}$ for $y > 0$? Using Exercises 1.3.15 and 1.3.16.
- ▶ How do I evaluate $\int_{s \in \mathbb{R}} \frac{c}{1+s^2}$? I need that twice, once for showing that g is absolutely integrable and once for evaluating the limit. By linearity $\int_{s \in \mathbb{R}} \frac{c}{1+s^2} = c \int_{s \in \mathbb{R}} \frac{1}{1+s^2}$. By Exercise 1.3.17 (Compatibility with the Riemann integral) $\int_{s \in \mathbb{R}} 1_{[-n,n]} \frac{1}{1+s^2} = \int_{-n}^n \frac{1}{1+s^2} ds$, where the integral on the left is the Lebesgue integral and the integral on the right is the Riemann integral. By the Fundamental Theorem of Calculus $\int_{-n}^n \frac{1}{1+s^2} ds = \arctan(n) - \arctan(-n) = 2 \arctan(n)$. $\int_{s \in \mathbb{R}} \frac{1}{1+s^2} = \lim_{n \rightarrow \infty} \int_{s \in \mathbb{R}} 1_{[-n,n]} \frac{1}{1+s^2} = 2 \lim_{n \rightarrow \infty} \arctan(n) = \pi$. The first of those equations comes either from Exercise 1.3.10(ix) (Vertical truncation) or from 1.4.43 (Monotone Convergence Theorem). It *doesn't* come from the Lebesgue Dominated Convergence Theorem. That would be circular.