MAU 22200 Week 8 Lecture 1

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Reading

For this week read

- Subsection 1.4.4 Measurable functions, and integration on a measure space
- Subsection 1.4.5 The Convergence Theorems

Exam information

I'll have to give you some sort of practice exam. Last year there was a final assignment in place of an exam and the years before that were invigilated 2 hour exams, so there isn't really a comparable exam to point you towards. I'll have to finish the real exam before I post a practice exam though.

The exam problems to be somewhat similar to the assignment problems, but easier and with more of an emphasis on examples than on theorems. And, of course, you won't be doing them in groups. Some problems may be taken directly from the book, but I haven't decided yet.

Comments on exercises

I'm more or less caught up on commenting on the solutions which have been posted. Sorry for the delay on some of them. I know your time is limited, but reading other groups' solutions, and possibly asking about gaps, may save you time in the end. I'll also give some continuous assessment credit for interesting comments or questions, especially if you have suggestions for how to fill any gaps you identify.

Strategy 2.1.5

Uncountable unions can sometimes be replaced by countable or finite unions.
Why would you want to do that? Countable unions of measurable sets are measurable. Countable unions of null sets are null sets.
An uncountable union of measurable sets needn't be measurable.
An uncountable union of null sets needn't be null, or even measurable.
For an example of where this is useful, consider proving that if a measurable unsigned function has integral 0 then it's 0 a.e. This is 1/4 of Exercise 1.3.18. Markov's inequality (Lemma 1.3.15)

$$m\left(\left\{x \in \mathbf{R}^d : f(x) \geq \lambda\right\}\right) \leq \frac{1}{\lambda} \int_{\mathrm{R}^d} f(x)\, dx$$

implies $\int_{\mathrm{R}^d} f(x)\, dx = 0 \Rightarrow m\left(\left\{x \in \mathbf{R}^d : f(x) \geq \lambda\right\}\right) = 0$ for all $\lambda > 0$.

Strategy 2.1.5 (continued)

$$\left\{x \in \mathbf{R}^d : f(x) \neq 0\right\} = \bigcup_{\lambda > 0} \left\{x \in \mathbf{R}^d : f(x) \geq \lambda\right\},$$

so this is a union of null sets. Unfortunately there are uncountably many positive real λ, so we can't immediately conclude that $\left\{x \in \mathbf{R}^d : f(x) \neq 0\right\}$ is a null set.
The way around this is to observe that

$$\left\{x \in \mathbf{R}^d : f(x) \neq 0\right\} = \bigcup_{n \in \mathrm{N}} \left\{x \in \mathbf{R}^d : f(x) \geq \frac{1}{n}\right\}.$$

There are only countably many $n \in \mathbf{N}$, so now we can conclude that $\left\{x \in \mathbf{R}^d : f(x) \neq 0\right\}$ is a null set. In other words, $f(x) = 0$ almost everywhere, as claimed.
This sort of trick is used all the time, but it's not always this easy to spot.

Uniqueness Theorems

There are a number of uniqueness theorems in the book.

- Exercise 1.1.3 (Elementary measure)
- Exercise 1.1.15 (Jordan measure)
- Exercise 1.2.23 (Lebesgue measure)
- Exercise 1.3.14 (Lebesgue integral)

Those are the ones we've seen but there will be a few others.

Ex 1.3.14 The Lebesgue integral $f \mapsto \int_{\mathrm{R}^d} f(x)\,dx$ is the only map from measurable unsigned functions to $[0, +\infty]$ such that:

- If f is simple then $\int_{\mathrm{R}^d} f(x)\,dx = \mathrm{Simp} \int_{\mathrm{R}^d} f(x)\,dx$.
- $\int_{\mathrm{R}^d}(f(x) + g(x))\,dx = \int_{\mathrm{R}^d} f(x)\,dx + \int_{\mathrm{R}^d} g(x)\,dx$.
- $\lim_{n\to\infty} \int_{\mathrm{R}^d} \min(f(x), n)\,dx = \int_{\mathrm{R}^d} f(x)\,dx$.
- $\lim_{n\to\infty} \int_{\mathrm{R}^d} f(x) 1_{|x|\leq n}\,dx = \int_{\mathrm{R}^d} f(x)\,dx$.

Interpretation

This phrasing things has the advantage that the properties are recognisably those of Exercise 1.3.10 and Corollary 1.3.14. It can be rephrased, though, to avoid confusing the Lebesgue integral with the mapping we're meant to show is the Lebesgue integral:

Suppose that Φ is a function from the set of measurable unsigned functions to the extended non-negative reals such that

- if f is simple then $\Phi(f) = \mathrm{Simp} \int_{\mathbf{R}^d} f(x)\, dx$,
- $\Phi(f+g) = \Phi(f) + \Phi(g)$,
- $\lim_{n\to\infty} \Phi(\min(f, n)) = \Phi(f)$, and
- $\lim_{n\to\infty} \Phi(f 1_{|x|\le n}) = \Phi(f)$.

Then $\Phi(f) = \int_{\mathbf{R}^d} f(x)\, dx$.

It's generally safer to write things this way. You're less tempted to use properties of the Lebesgue integral on Φ while you're still trying to prove that $\Phi(f)$ is the Lebesgue integral of f.

Uses

There's more than intellectual curiosity motivating such theorems. They're useful for proving identities.
Exercise 1.3.15 asks you to prove that if $f \colon \mathbf{R}^d \to [0, +\infty]$ is measurable then $\int_{\mathrm{R}^d} f(x+y)\,dx = \int_{\mathrm{R}^d} f(x)\,dx$ for any $y \in \mathbf{R}^d$.
This can be generalised to: If $f \colon \mathbf{R}^d \to [0, +\infty]$ is measurable $h \colon \mathbf{R}^d \to \mathbf{R}^d$ is an invertible function such that $m(h(E)) = m(E)$ for all measurable sets E then $\int_{\mathrm{R}^d} f(h(x))\,dx = \int_{\mathrm{R}^d} f(x)\,dx$.
One way to prove this is to let $\Phi(f) = \int_{\mathrm{R}^d} f(h(x))\,dx$ and then prove that Φ has the properties above. I'll give most of the proof now.

Proof (1/3)

Let $\Phi(f) = \int_{\mathrm{R}^d} f(h(x))\,dx$.

$$1_E(h(x)) = 1 \Leftrightarrow h(x) \in E \Leftrightarrow x \in h^{-1}(E) \Leftrightarrow 1_{h^{-1}(E)}(x) = 1$$

so $1_E \circ h = 1_{h^{-1}(E)}$. If f is simple, i.e. $f = \sum_{j=1}^k c_j 1_{E_j}$ then $f \circ h = \sum_{j=1}^k c_j 1_{E_j} \circ h = \sum_{j=1}^k c_j 1_{h^{-1}(E_j)}$ so

$$\Phi(f) = \int_{\mathrm{R}^d} f(h(x))\,dx = \int_{\mathrm{R}^d} \left(\sum_{j=1}^k c_j 1_{h^{-1}(E_j)}(x) \right)\, dx$$

$$= \sum_{j=1}^k c_j m(h^{-1}(E_j)) = \sum_{j=1}^k c_j m(E_j)$$

$$= \mathrm{Simp} \int_{\mathrm{R}^d} f(x)\,dx.$$

Proof (2/3)

$$\begin{aligned}\Phi(f+g) &= \int_{\mathrm{R}^d} (f+g)(h(x))\,dx = \int_{\mathrm{R}^d} (f(h(x)) + g(h(x)))\,dx \\ &= \int_{\mathrm{R}^d} f(h(x))\,dx + \int_{\mathrm{R}^d} g(h(x))\,dx = \Phi(f) + \Phi(g).\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \Phi(\min(f,n)) &= \lim_{n\to\infty} \int_{\mathrm{R}^d} \min(f(h(x)), n)\,dx \\ &= \int_{\mathrm{R}^d} f(h(x))\,dx = \Phi(f).\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \Phi(f 1_{|x|\le n}) &= \lim_{n\to\infty} \int_{\mathrm{R}^d} f(h(x)) 1_{|x|\le n}(h(x))\,dx \\ &= \lim_{n\to\infty} \int_{\mathrm{R}^d} f(h(x)) 1_{|h(x)|\le n}(x)\,dx.\end{aligned}$$

Proof (3/3)

If we can show that

$$\lim_{n\to\infty} \int_{\mathrm{R}^d} f(h(x)) 1_{|h(x)|\le n}(x)\, dx = \int_{\mathrm{R}^d} f(h(x))\, dx.$$

then $\lim_{n\to\infty} \Phi(f 1_{|x|\le n}) = \Phi(f)$ and Φ satisfies all the hypotheses of the uniqueness theorem and therefore $\Phi(f) = \int_{\mathrm{R}^d} f(x)\, dx$, i.e.

$$\int_{\mathrm{R}^d} f(h(x))\, dx = \int_{\mathrm{R}^d} f(x)\, dx.$$

Unfortunately we don't have a theorem yet which shows this. You can adapt the proof of Exercise 1.3.10(ix) to give it. Alternatively, you can wait and use the Monotone Convergence Theorem (Theorem 1.4.43). This isn't the only way to prove that $\int_{\mathrm{R}^d} f(h(x))\, dx = \int_{\mathrm{R}^d} f(x)\, dx$, but it works, and illustrates how uniqueness theorems can be used to prove identities.