MAU 22200 Week 6 Lecture 1

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8 March 2021

Reading

For this week read

- Introduction to Section 1.4 Abstract Measure Spaces
- Subsection 1.4.1 Boolean algebras
- Subsection 1.4.2 σ algebras and measurable spaces
- Subsection 1.4.3 Countably additive measures and measure spaces

Littlewood's Three Principles (1/4)

There are various theorems which say that the set where bad things happen is small. There's usually a trade-off between how bad you allow things to be on the exceptional set and how large you allow the exceptional set to be. For example:

- Exercise 1.2.16 Every measurable set of finite measure differs from an open/compact/bounded/elementary set by a set of arbitrarily small measure.
- Exercise 1.2.19 Every measurable set differs from countable intersection of open sets or a countable union of closed sets by a set of measure 0.

Those are theorems for sets, but there are also theorems for functions or sequences of functions.

Littlewood's Three Principles (2/4)

For example:

- Lusin's Theorem (1.3.28) Let $f\colon \mathbf{R}^d \to \mathbf{C}$ be absolutely integrable and $\epsilon > 0$. Then there exists a Lebesgue measurable $E \subset \mathbf{R}^d$ such that $m(E) < \epsilon$ and the restriction of f to $\mathbf{R}^d \backslash E$ is continuous.
- Egorov's Theorem (1.3.26) Let $f_n\colon \mathbf{R}^d \to \mathbf{C}$ be a sequence of measurable functions converging pointwise almost everywhere to f and $\epsilon > 0$. Then there exists a Lebesgue measurable A such that $m(A) < \epsilon$ and f_n converges locally uniformly to f on $\mathbf{R}^d - A$.

You have to be careful to interpret such statements correctly. For example, saying f restricted to C is continuous is not the same as saying that f is continuous at all points in C. $1_{[0,+\infty)}$ restricted to $[0, +\infty)$ is continuous, for example, but $1_{[0,+\infty)}$ is not continuous at the point $0 \in [0, +\infty)$.

Littlewood's Three Principles (3/4)

Countable additivity is needed to prove such results, which is why the Lebesgue theory is more useful than the Jordan theory. For example, here's a proof of Lusin's Theorem:
Every absolutely integrable function can be approximated arbitrarily well, in the $L^1(\mathbf{R}^d)$ sense, by a compactly supported continuous function (Theorem 1.3.20). So there's a compactly supported continuous f_n with $\int |f(x) - f_n(x)|\, dx < \epsilon/4^n$. Markov's inequality (Lemma 1.3.15) shows that $m(E_n) < \epsilon/2^n$ where $E_n = \{x \in \mathbf{R}^d \colon |f(x) - f_n(x)| \geq 1/2^n\}$. $E = \bigcup_{n \in \mathrm{N}} E_n$ is measurable and $m(E) < \epsilon$. This is where we use countable additivity (explicitly). On $\mathbf{R}^d \backslash E$ we have $|f_n(x) - f(x)| \leq 1/2^n$ for all n, so the restriction of f_n to $\mathbf{R}^d \backslash E$ converges to the restriction of f uniformly. Uniform limits of sequences of continuous functions are continuous, so the restriction of f is continuous.

Littlewood's Three Principles (4/4)

Roughly speaking,

1. Every measurable set is nearly elementary.
2. Every absolutely integrable function is nearly continuous.
3. Every pointwise convergent sequence is nearly uniformly convergent.

These are known as *Littlewood's three principles* He stated them in his 1944 textbook in a slightly different form. For example, in his book $d = 1$, so he writes "a finite union of intervals" in place of "elementary".

These aren't theorems, so you can't cite them to justify an argument, but they are helpful in finding an argument. These are related to Strategy 2.1.7 (Be willing to throw away an exceptional set) and Strategy 2.1.9 (Try simpler cases first). There are a variety of theorems corresponding to each of the principles. The choice of which one to use tends to be dictated by the conclusion of the result you're trying to prove.

Abstract measure and integration

Some, but not all, of the Lebesgue theory can be generalised beyond $\mathbf{R}^d$. Far beyond $\mathbf{R}^d$.
In Statistical Mechanics or Financial Mathematics one considers all possible trajectories of a particle or of an asset price and assigns probabilities and takes expected values. The probabilities can be interpreted as measures and the expected values as integrals. These are measures or integrals of functions on the set of all trajectories. There's also path-integral formulation of Quantum Mechanics. The abstract version of measure and integration covers the Statistical Mechanics and Financial Mathematics applications, but not really the Quantum Mechanics one.
The abstract theory is also useful for understanding the special case of Lebesgue integration.

What do you need?

What are the main ingredients of Lebesgue integration?

- A set X. In Lebesgue integration $X = \mathbf{R}^d$.
- A set $\mathcal{B}$ of subsets of X which includes the empty set and is closed under complements, countable unions and countable intersections. In Lebesgue integration $\mathcal{B}$ is the set of Lebesgue measurable subsets.
- A function μ from $\mathcal{B}$ to $[0, +\infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n \in \mathrm{N}} E_n) = \sum_{n \in \mathrm{N}} \mu(E_n)$ if the E's are disjoint elements of $\mathcal{B}$. In Lebesgue integration $\mu = m$.

You can do most of the theory of integration with any $(X, \mathcal{B}, \mu)$ with the properties above. There are some exceptions. There's no concept of bounded, open, compact, etc. in such an X. So you can't expect a result like Lusin's Theorem, unless X has a topology and $\mathcal{B}$ and μ satisfy additional hypotheses. You can expect some version of Egorov's Theorem though.

What is and isn't in Section 1.4

Section 1.4 is about defining integrals and proving their properties given a triple $(X, \mathcal{B}, \mu)$ as above.

That's only half of what we did in the Lebesgue case. The other half was constructing a particular $\mathcal{B}$ and μ, the Lebesgue measurable sets and the Lebesgue measure. Those were constructed using the structure $\mathbf{R}^d$. E.g. a measurable set is one which can be approximated from without by an open set, with an error contained in a countable union of boxes whose total size can be made arbitrarily small.

Section 1.4 just assumes you have a triple $(X, \mathcal{B}, \mu)$. It says very little about how you might construct one. That's discussed in Section 1.7.

Section 1.4 isn't just generalising results from Section 1.3 though. Subsection 1.4.5 is full of useful theorems which you haven't yet seen, even in the Lebesgue case.