

MAU 22200 Week 5 Lecture 2

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Most of this extends to real or complex valued functions. For real valued f you'd want both $\{(x, t) \in \mathbf{R}^{d+1}: 0 \leq t \leq f(x)\}$ and $\{(x, t) \in \mathbf{R}^{d+1}: f(x) \leq t \leq 0\}$ to be Lebesgue measurable.

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What can you use upper and lower integrals for?

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Unsigned vs real or complex

Unsigned simple integrals are used to define unsigned integrals in of measurable unsigned functions. (1.3.13)

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$$f(x) = f_+(x) - f_-(x)$$

$$f_+(x) = \max(f(x), 0)$$

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By definition $\int f = \int f_+ - \int f_-$. Absolute integrability is needed for this subtraction to make sense. Complex integrals are defined in terms of real integrals.

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