# MAU 22200 Week 5 Lecture 1

John Stalker

Trinity College Dublin

1 March 2021

# Reading

For this week, read the rest of Section 1.3, i.e.

- 1.3.3 Unsigned Lebesgue integrals
- 1.3.4 Absolute integrability
- 1.3.5 Littlewood's three principles

# What facts can you use in doing exercises?

- What facts from lectures are you allowed to use in doing exercises? Not many. Most of those facts are restatements of results from the book. The rules for those are the ones from Week 2 Lecture 1. Roughly, you can use them if they appear before the exercise you're doing. Also, you can use non-exercises that appear later, if there's no risk of circularity. You can also use anything that appears later, if you prove it.
- What about facts from lecture which don't appear in the book? You can use any of the cardinality and countability results. You can't use Banach-Tarski, but you probably wouldn't want to.
- You can use anything from previous semesters, in this module or others.

# Terminology

- Agree: are equal. E.g. "$f$ is Darboux integrable if its upper and lower integrals agree." Or, $1_{(-1,1)}$ and $1_{[-1,1]}$ agree except at $\pm 1$.
- Almost everywhere: Except on a set of (Lebesgue) measure zero. E.g If $E$ is Jordan measurable then $1_{E^\circ} = 1_{\overline{E}}$ almost everywhere.
- Agree almost everywhere: Are equal except on a (possibly empty) set of measure zero. E.g If $E$ is Jordan measurable then $1_{E^\circ}$ and $1_{\overline{E}}$ agree almost everywhere.
- Relatively open or relatively closed: Open or closed in the subspace topology. E.g. As subsets of $[-1,1)$, $[-1,0)$ is relatively open, while $[0,1)$ is relatively closed.

# How discontinuous can a function be and still be Riemann integrable?

**Exercise 1.1.23** Show that any continuous function $f : [a, b] \to \mathbf{R}$ is Riemann integrable. More generally, show that any bounded, piecewise continuous function $f : [a, b] \to \mathbf{R}$ is Riemann integrable.

The boundedness assumption is necessary, but is redundant in the piecewise continuous case. It is *not* redundant in the piecewise linear case. $f : [-1, 1] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1/x & \text{if } x > 0 \end{cases}$$

is piecewise continuous, but not bounded.

Exercise 1.1.23 says that a bounded function on a closed interval is Riemann integrable if its set of points of discontinuity is finite.

# The number of discontinuities can be infinite

Note that it doesn't say that a bounded function on a closed interval is Riemann integrable if and only if its set of points of discontinuity is finite. Just how large can the set of points of discontinuity be?
It can definitely be infinite. Define $f(x) = 0$ if $x$ is irrational and $f(x) = 1/q$ if $x$ is rational and $q$ is the smallest positive integer such that $qx$ is an integer. One can show that

- $f$ is discontinuous at every rational number and continuous at every irrational number.
- $f$ is Riemann integrable on every closed interval.

The set of points of discontinuity can even be uncountable. $1_C$ where $C$ is the Cantor set from Exercise 1.2.9 is Riemann integrable but is discontinuous at every point of $C$.

# A theorem of Lebesgue

Lebesgue proved that a bounded function on a closed interval is Riemann integrable if and only if its set of points of discontinuity is of Lebesgue measure 0. I'll prove the "if" part.
Suppose $f : [a, b] \to \mathbf{R}$ is bounded, $D \subset [a, b]$, $m(D) = 0$ and $f$ is continuous at all $x \notin D$.
Let $B_{x,\delta} = [a, b] \cap (x - \delta, x + \delta)$. $B_{x,\delta}$ is relatively open.
Define the oscillation of $f$ in a set $J \subset I$ by

$$O(f, J) = \sup_J f - \inf_J f.$$

This is always finite because $f$ is bounded. If $K \subset J$ then $O(f, K) \leq O(f, J)$. Oscillation is important because if $J_1, \ldots, J_n$ is a partition $[a, b]$ into disjoint intervals then

$$\overline{\int_a^b} f(x)\, dx \leq \underline{\int_a^b} f(x)\, dx + \sum_{i=1}^{n} O(f, J_i)|J_i|$$

# Oscillation at a point and continuity

Define the oscilation of $f$ at $x \in [a, b]$ by

$$o(f, x) = \inf_{\delta > 0} O(f, B_{x,\delta}).$$

It's non-negative. If $f$ is continuous at $x$ and $\kappa > 0$ then $\frac{\kappa}{2} > 0$ so there is a $\delta > 0$ such that $y \in B_{x,\delta} \Rightarrow |f(y) - f(x)| < \frac{\kappa}{2}$. Therefore $\sup_{B_{x,\delta}} f \leq f(x) + \frac{\kappa}{2}$ and $\inf_{B_{x,\delta}} f \geq f(x) - \frac{\kappa}{2}$. So $O(f, B_{x,\delta}) \leq \kappa$ and $o(f, x) \leq \kappa$. This holds for all $\kappa > 0$, so $o(f, x) = 0$.
For any $\kappa > 0$, the set

$$C_\kappa = \{x \in [a, b] \colon o(f, x) < \kappa\}$$

is open. To see this, note that if $x \in C_\kappa$ then there is a $\delta > 0$ such that $O(f, B_{x,\delta}) < \kappa$. If $y \in B_{x,\delta/2}$ then $B_{y,\delta/2} \subset B_{x,\delta}$ and so $o(f, y) < \kappa$ and $y \in C_\kappa$.

# Constructing a finite open cover

For $\epsilon > 0$, let $\kappa = \frac{\epsilon}{3(b-a)+1}$ and $\lambda = \frac{\epsilon}{3O(f,[a,b])+1}$. Then $\kappa, \lambda > 0$. Let $K = [a,b]\backslash C_\kappa$. $K$ is the intersection of the compact set $[a,b]$ and the closed set $C_\kappa^c$, so it's compact. If $x \in K$ then $o(f,x) \geq \kappa > 0$ so $f$ is not continuous at $x$ and $x \in D$. $K \subset D$ and $m(D) = 0$ so $m(K) = 0$ and therefore $m^*(K) = 0$. There are therefore intervals $I_1$, $I_2$, ...such that $K \subset \bigcup_{i=1}^\infty I_i$ and $\sum_{i=1}^\infty |I_i| < \lambda$. Let $\Omega_i$ be the open interval with the same midpoint as $I_i$ and $|\Omega_i| = |I_i| + \frac{\lambda}{2^i}$. Then $I_i \subset \Omega_i$ and therefore $K \subset \bigcup_{i=1}^\infty \Omega_i$. Also, $\sum_{i=1}^\infty |\Omega_i| = \sum_{i=1}^\infty |I_i| + \sum_{i=1}^\infty \frac{\lambda}{2^i} < 2\lambda$. $K$ is compact, so there's a finite subcover, i.e. $K \subset \Omega$ where $\Omega = \bigcup_{j=1}^m \Omega_{i_j}$. Also, $\Omega$ is open and $\sum_{j=1}^m |\Omega_{i_j}| < 2\lambda$. Let $L = [a,b]\backslash\Omega$. $L$ is a compact subset of $C_\kappa$. For each $x \in L$ there's a $\delta > 0$ such that $B_{x,\delta} \subset C_\kappa$. That's an open cover of $L$. Choose a finite subcover. $B_{x_1,\delta_1}, \ldots, B_{x_n,\delta_n}$. Let $\Omega_{m+j} = B_{x_j,\delta_j}$.

# Constructing a partition

The open intervals $\Omega_1, \ldots \Omega_{m+n}$ which cover $[a, b]$. Any finite union of boxes can be written as a finite union of disjoint boxes, where each box in the second collection is a subset of a box in the first collection (see Lemma 1.1.2). Let $J_1, \ldots, J_p$ be those contained in $\Omega_j$, $j \le m$. $\sum_{i=1}^{p} |J_i| \le \sum_{j=1}^{m} |\Omega_j| < 2\lambda$. Let $J_{p+1}, \ldots, J_{p+q}$ be the remaining intervals, which are subsets of some $\Omega_j$ with $j > m$ and hence satisfy $O(f, J_i) \le O(f, \Omega_j) < \kappa$.

$$\overline{\int_a^b} f(x)\, dx \le \underline{\int_a^b} f(x)\, dx + \sum_{i=1}^{p} O(f, J_i)|J_i| + \sum_{i=p+1}^{p+q} O(f, J_i)|J_i|$$

$$\sum_{i=1}^{p} O(f, J_i)|J_i| \le O(f, [a, b]) \sum_{i=1}^{p} |J_i| \le 2\lambda O(f, [a, b]) < \frac{2\epsilon}{3},$$

$$\sum_{i=p+1}^{p+q} O(f, J_i)|J_i| \le \kappa \sum_{i=p+1}^{p+q} |J_i| \le \kappa(b - a) < \frac{\epsilon}{3}$$

# Conclusion

We now have

$$\overline{\int_a^b} f(x)\,dx < \underline{\int_a^b} f(x)\,dx + \epsilon$$

for all $\epsilon > 0$, and hence

$$\overline{\int_a^b} f(x)\,dx \leq \underline{\int_a^b} f(x)\,dx$$

The reverse inequality holds for all bounded functions, so

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

So $f$ is Riemann integrable.