MAU 22200 Week 4 Lecture 2

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Comments on constructing proofs (1/7)

It's easy to miss the fact that an argument fails in a trivial case. Here's an example which arises in Riemann/Darboux integration of continuous functions.

You have a continuous function g on an interval $[a, b]$ and want to show it's integrable, by showing that the lower and upper integrals are equal. You split the equality up into two equalities (2.1.1). In other words, you try to show

$$\underline{\int_a^b} g(x)\, dx \le \overline{\int_a^b} g(x)\, dx, \qquad \overline{\int_a^b} g(x)\, dx \le \underline{\int_a^b} g(x)\, dx.$$

The first inequality is easy. The second, not so much. To prove the second you give yourself an ϵ of room (2.1.2). In other words, you try to prove that for all $\epsilon > 0$

$$\overline{\int_a^b} g(x)\, dx \le \underline{\int_a^b} g(x)\, dx + \epsilon.$$

Because of how upper and lower sums are defined,

$$\overline{\int_a^b} g(x)\, dx \leq \underline{\int_a^b} g(x)\, dx + \epsilon.$$

will follow if there are piecewise continuous functions f and h on $[a, b]$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$ and

$$\int_a^b h(x)\, dx \leq \int_a^b f(x)\, dx + \epsilon.$$

This in turn will follow if $(b - a)(h(x) - f(x)) \leq \epsilon$ for all $x \in [a, b]$. That would follow from $(b - a)(h(x) - g(x)) \leq \frac{\epsilon}{2}$ and $(b - a)(g(x) - f(x)) \leq \frac{\epsilon}{2}$ for all $x \in [a, b]$.

If $I \subset [a, b]$ then
$$\min_I g \leq g(x) \leq \max_I g$$
for $x \in I$ so we try partitioning $[a, b]$ into subintervals $I_1, \ldots I_n$ and defining $f(x) = \min_{\bar{I_j}} g$ for $x \in I_j$ and $h(x) = \max_{\bar{I_j}} g$ for $x \in I_j$. For this to work we need $(b - a)(\max_{\bar{I_j}} g - g(x)) \leq \frac{\epsilon}{2}$ and $(b - a)(g(x) - \min_{\bar{I_j}} g) \leq \frac{\epsilon}{2}$ for $x \in I_j$. That works if I_j is small enough. More precisely, continuous functions on closed intervals are uniformly continuous. In other words, for any $\epsilon' > 0$ there is a $\delta > 0$ such that if $|x - y| < \delta$ then $|g(x) - g(y)| < \epsilon'$. Note that I had to call it ϵ' rather than ϵ because we already have an ϵ. If we take ϵ' small enough that $(b - a)\epsilon' \leq \frac{\epsilon}{2}$ and we take the δ which corresponds to it, as above, then we'll have $\max_{\bar{I_j}} g - g(x) \leq \epsilon'$ and therefore $(b - a)(\max_{\bar{I_j}} g - g(x)) \leq \frac{\epsilon}{2}$ in any interval I_j of length less than δ, and similarly $(b - a)(g(x) - \min_{\bar{I_j}} g) \leq \frac{\epsilon}{2}$.

How many intervals do we need? If we split $[a, b]$ into n intervals of equal length then they'll each have length $(b - a)/n$, which we need to be less than δ.

We now have a proof, more or less. If we write it like this then it's easy to see how we got it, but it's hard to read. Usually we rearrange the order, to make it easier to follow.

A more usual way to write this would start with something like "For every $\epsilon > 0$ set $\epsilon' = \frac{\epsilon}{2(b-a)}$. g is uniformly continuous and $\epsilon' > 0$, so there is a $\delta > 0$ such that $|x - y| < \delta \Rightarrow |g(x) - g(y)| < \epsilon'$. Choose an $n > (b - a)/\delta$. Define $I_j = [a + (j - 1)(b - a)/n, a + j(b - a)/n)$ for $j < n$ and $I_n = [a + (n - 1)(b - a)/n, b]$. Then $[a, b]$ is the disjoint union of $I_1, \ldots I_n$. Define $f(x) = \min_{\bar{I}_j} g$ and $h(x) = \max_{\bar{I}_j} g$ for $x \in I_j$. ..."

I'm not going to continue the rearrangement, since I have enough of it to illustrate all the points I want to make.

There are several things I did without mentioning them specifically, to avoid possible problems. I treated I_n differently from I_j for $j < n$. The points where we switch from one subinterval to the next have to be in or the other of those subintervals, not both, in order to make f and h well-defined. I chose to put such points in the subinterval to their right, not the subinterval to their left. But we need b to be included in some subinterval, so the last subinterval needs to be treated differently. If I had made the opposite choice then it would be the first interval I'd need to treat separately. Since my subintervals aren't all closed I took maxima and minima over their closures. The maximum or minimum over I_j might not exist.

Comments on constructing proofs (6/7)

Is this actually easier to read? Yes and no. It's easier to check line by line than an argument written in reverse order. It's easy to see why we're allowed to do what we're doing at each step, but it's often unclear why we'd want to.

I described the construction of the proof working backwards from the conclusion to the hypotheses. More often than not, that's how I construct proofs, even though the clean version will be written in the reverse order. You don't have to do that. I don't think Terry does, most of the time. If you start from the hypotheses though then you often have quite a few numbers, sets, functions, etc. to be chosen later, and will need to make sure they can be chosen in the correct order so as to have the required properties. That mostly works when you know the outline of the proof in advance.

There's something I've been concealing from you. The proof fragment from slide 4 has an error in its first sentence! "For every $\epsilon > 0$ set $\epsilon' = \frac{\epsilon}{2(b-a)}$." Who said $a \neq b$? We could be dividing by zero. All we know about the interval $[a, b]$ is that $a \leq b$. Allowing single points to be intervals is useful, although it creates a minor problem here.

How do we fix this? One option is to treat the trivial case separately. It's easy, so why not? That's fine here but the number of special cases to be considered grows exponentially in the number of things which could be trivial. There's usually a better way. Here the trick is to choose $\epsilon' = \frac{\epsilon}{2(b-a)+1}$. You still get $(b-a)\epsilon' \leq \frac{\epsilon}{2}$, which is all you need, and the denominator can't be zero.