MAU 22200 Week 4 Lecture 1

John Stalker

Trinity College Dublin

22 February 2021

Apologies

- I didn't email last week's problem and group assignments until Tuesday. I'll try to keep them to Mondays.
- My internet was slow, so slides and video for Thursday's lecture appeared on the module webpage a few minutes late. I'll try to avoid that.
- Blackboard (really Panopto) has a preprocessing step for uploaded video. It can take hours. I try to upload lectures to Blackboard the night before, like this one, but that won't always be possible. You don't have to watch prerecorded lectures at the exact time they're timetabled, but it can be a good way of not falling behind. If you want to do that then you're better off with the webpage than with Blackboard, unless you actually need its additional functionality.
- I forgot to send out the meeting invite for Friday's Q&A until just before. I'll try to send the rest out earlier.

Reading for this week (and last week)

I forgot to indicate which subsections to read for last week, but 1.2.2 (Lebesgue measurability) would be would be good to read for that, if you haven't already.

For this week you should read

- 1.2.3 (Non-measurable sets)
- The introduction to 1.3 (The Lebesgue integral)
- 1.3.1 (Integration of simple functions)
- 1.3.2 (Measurable functions)

Highlights from Section 1.2

- Definition (unnumbered) of Lebesgue outer measure in terms of approximation from without by countable unions of boxes.
- Definition 1.2.2 (definition of Lebesgue measurability, and Lebesgue measure, in terms of approximation from within by open sets and Lebesgue outer measure) Approximate in the sense that the difference can be made to have arbitrarily small outer measure.
- Exercise 1.2.3 (empty set, monotonicity and countable subadditivity properties of Lebesgue outer measure)
- Remark 1.2.8 (even topologically nice bounded sets don't have to be Jordan measurable)
- Lemma 1.2.13 (Existence of Lebesgue measurable sets) Any topologically nice set is Lebesgue measurable) Countable unions and intersections of Lebesgue measurable sets are Lebesgue measurable.
- Exercise 1.2.7 (Criteria for measurability)

More highlights from Section 1.2

- Exercise 1.2.8 (every Jordan measurable set is Lebesgue measurable)
- Lemma 1.2.15 (empty set and countable additivity properties of Lebesgue measure) Also countable subadditivity.
- Exercise 1.2.11 (Monotone convergence theorem for measurable sets) This will be used repeatedly, and foreshadows a monotone convergence theorem for integrals.
- Exercise 1.2.16 (Criteria for finite measure)
- Exercise 1.2.20 (Translation invariance)
- Exercise 1.2.21 (Behaviour under linear transformations)
- Exercise 1.2.22 (Behaviour under Cartesian product)
- Exercise 1.2.23 (Uniqueness of Lebesgue measure)
- Proposition 1.2.18 (Not every set is Lebesgue measurable.)

Definition of Lebesgue integration (main ideas)

There are two simpler theories which are useful to keep in mind for Lebesgue integration: Summation and Riemann/Darboux integration.
There are really two different version of integration theory, one for non-negative extended real valued functions, sometimes called unsigned functions, and one for finite real valued functions. The second theory extends easily to complex valued functions or vector valued functions.
More or less the same thing happens for sums, where we talk about absolute convergence. Considering convergent but not absolutely convergent sums is usually a terrible idea. Lebesgue integration theory doesn't make the analogous mistake. Allowing functions to take infinite values is more useful than allowing sums to have infinite summands.

Definition of Lebesgue integration (more main ideas)

Darboux integration (Subsection 1.1.3, plus additional comments in Section 1.3) is based on approximation by piecewise constant functions. The “pieces” in the word “piecewise” refer to intervals. A piecewise constant function in this sense is one which takes constant values on each of finitely many pieces, and is zero everywhere else. Integrals of piecewise constant functions are easy to define and have nice properties, e.g. linearity, monotonicity, etc. Both the definition and the properties are then extended to more general functions, but not to all functions. Choosing intervals is mostly a matter of convenience. You’d get the same theory if you chose Jordan measurable sets as pieces. By this I mean that the same functions would end up being integrable, and they would have the same integrals. Some proofs would be harder while others would be easier.

Definition of Lebesgue integration (still more main ideas)

There are two obvious steps to take to get a more general theory of integration:

- Allow the Lebesgue measurable pieces rather than Jordan measurable.
- Allow countably many of them.

Oddly, it's technically easier to make only the first change. This leads to what are called simple functions. Integrals of simple functions are easy to define and have nice properties, e.g. linearity, monotonicity, etc. Both the definition and the properties are then extended to more general functions, but not to all functions. This extension is more complicated than for the Darboux integral though.