MAU 22200 Week 3 Lecture 2

John Stalker

Trinity College Dublin

18 February 2021

Comments on Exercises

- Groups are generally doing the exercises, and most solutions are correct.
- I've commented at least briefly on most of them.
- I've tried to flag any major problems, but I'm mostly ignoring any minor issues.
- Not all comments are corrections.
- You should read other groups' solutions to their exercises.
- It can be a useful source of ideas for exercises you'll have do later.
- Reading other people's solutions, which may or may not be correct, is also useful for understanding how to write your own.
- You should ask about anything that isn't clear to you.

How and why to write shorter solutions

- The shorter your solution is, the fewer opportunities you have to get something wrong.
- Also, the shorter it is, the easier it will be to find any errors you have made.
- Usually, but not always, a shorter argument is easier for other people to read.
- Try to avoid case by case analysis where you can. If you can't avoid it, try strategy 2.1.4.
- Definitely avoid redundant cases! If there's a trivial case and a non-trivial case then you can often modify the argument for the latter to cover the former.
- Don't reinvent the wheel. Use earlier theorems or exercises as much as you can.

What's so special about countability?

Why is **N** special?

The Archimedean property: For all $\epsilon > 0$ there is an $n \in \mathbf{N}$ such that $\frac{1}{n} < \epsilon$. Consequences include

- The density of **Q** in **R**
- All the sequence criteria for metric spaces
- Exercise 0.0.1: If $(x_\alpha)_{\alpha \in A}$ is a collection of numbers $x_\alpha \in [0, +\infty]$ such that $\sum_{\alpha \in A} x_\alpha < \infty$, show that $x_\alpha = 0$ for all but at most countably many $\alpha \in A$, even if A itself is uncountable.

To prove Exercise 0.0.1, let $S = \sum_{\alpha \in A} x_\alpha$, $P = \{\alpha \in A \colon x_\alpha > 0\}$ and $P_n = \left\{\alpha \in A \colon x_\alpha > \frac{1}{n}\right\}$. If $\alpha \in P$ then $x_\alpha > 0$ so there is an $n \in \mathbf{N}$ such that $\frac{1}{n} < x_\alpha$ and hence $\alpha \in P_n$. In other words, $P = \bigcup_{n \in \mathrm{N}} P_n$. $nS = \sum_{\alpha \in A} n x_\alpha \geq \sum_{\alpha \in P_n} n x_\alpha \geq \# P_n$ so $\# P_n \leq nS$. P is a countable union of finite sets, so is either finite or countable.

A “converse” to Exercise 0.0.1

Any positive number can be written as a sum of countably many positive numbers. For example, $\epsilon = \sum_{n \in \mathrm{N}} \frac{\epsilon}{2^n}$ or $\epsilon = \sum_{n \in \mathrm{n}} \frac{6\epsilon}{(n\pi)^2}$.
You can't write $\epsilon > 0$ as the sum of countably many copies of *the same* positive number, by the Archimedean property, but you can write it as the sum of countably many positive numbers. You couldn't write as the sum of more than countably many, by Exercise 0.0.1.
There's an “$\frac{\epsilon}{2}$ trick” which is frequently used for proving statements about finite sums, unions, etc. There's a corresponding “$\frac{\epsilon}{2^n}$ trick” which can often be used for proving similar statements about countable sums, unions, etc. See Example 1.2.1, Exercise 1.2.3, Remark 1.2.7, Remark 1.2.8, Exercise 1.2.7 and Lemma 1.2.15 for examples.

Countable subadditivity of Lebesgue outer measure

There are boxes $B_{m,1}, B_{m,2}, \ldots$ such that $E_m \subset \bigcup_{n\in\mathbb{N}} B_{m,n}$ and

$$\sum_{n\in\mathbb{N}} m(B_{m,n}) < m^*(E_m) + \frac{\epsilon}{2^m}.$$

Then $\bigcup_{m\in\mathbb{N}} E_m \subset \bigcup_{(m,n)\in\mathbb{N}^2} B_{m,n}$ and

$$\sum_{(m,n)\in\mathbb{N}^2} m(B_{m,n}) < \sum_{m\in\mathbb{N}} m^*(E_m) + \epsilon.$$

It follows that

$$m^*\left(\bigcup_{m\in\mathbb{N}} E_n\right) < \sum_{m\in\mathbb{N}} m^*(E_m) + \epsilon.$$

for all $\epsilon > 0$ and so

$$m^*\left(\bigcup_{m\in\mathbb{N}} E_n\right) \leq \sum_{m\in\mathbb{N}} m^*(E_m).$$