MAU 22200 Week 3 Lecture 1

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15 February 2021

Cardinality

The number of elements of a set is called its *cardinality*. The book uses the notation $\#A$ for the cardinality of a set A.

I'll mostly just write $\#A = \infty$ if A is infinite, but there's a whole theory of cardinal and ordinal numbers, and there are different types of infinity. Advanced books on set theory describe these in detail, but we only really need $\#\mathbf{N}$ and $\#\mathbf{R}$ in this module.

We say that $\#A \leq \#B$ if there is an injective function from A to B. We say that $\#A \geq \#B$ if there is a surjective function from A to B. We'll say that $\#A = \#B$ if there is a bijective function from A to B. These have all the properties you would expect.

Properties of Cardinality

- $\#A \leq \#B \Rightarrow \#B \geq \#A$.
- $\#A \geq \#B \Rightarrow \#B \leq \#A$.
- $\#A = \#B \Rightarrow \#B = \#A$.
- $\#A \leq \#B \text{ and } \#B \leq \#C \Rightarrow \#A \leq \#C$.
- $\#A \leq \#B \text{ and } \#C \leq \#D \Rightarrow \#(A \times C) \leq \#(B \times D)$.
- $\#A = \#B \Rightarrow \#A \leq \#B \text{ and } \#B \leq \#A$.
- $\#A \leq \#B \text{ and } \#B \leq \#A \Rightarrow \#A = \#B$.

The first six are all immediate consequences of the definition, while the last is the Schröder-Bernstein Theorem.
If $A = B$ then $\#A = \#B$ and if $A \subset B$ then $\#A \leq \#B$. The converse isn't true, of course.
For *finite* sets B, if $A \subset B$ and $\#B = \#A$ then $B = A$. This is actually how finite is defined in set theory.

Countable sets

There is a bijection $i\colon \mathbf{N} \to 2\mathbf{N}$, defined by $i(n) = 2n$, so $\#\mathbf{N} = \#(2\mathbf{N})$. So $2\mathbf{N} \subset \mathbf{N}$ and $\#\mathbf{N} = \#(2\mathbf{N})$ but $\mathbf{N} \neq 2\mathbf{N}$. So $\mathbf{N}$ is not finite.

If $\#\mathbf{N} = \#A$ then we say that A is *countable*. For example $2\mathbf{N}$ is countable. So is $\mathbf{Z}$, since $j\colon \mathbf{N} \to \mathbf{Z}$, defined by

$$j(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

is a bijection. $\mathbf{N} \times \mathbf{N}$ is also countable, since $k\colon \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ defined by $k(m,n) = (m+n-1)(m+n-2)/2 + m$ is a bijection. To see where this odd looking function comes from, try making a table with $k(m,n)$ in the m'th row, n'th column.

$\mathbf{Q}$ is countable

For more complicated sets, we use what we already have.
$f(m, n) = m/n$ is a surjection from $\mathbf{Z} \times \mathbf{N}$ to $\mathbf{Q}$, so $\#(\mathbf{Z} \times \mathbf{N}) \geq \#\mathbf{Q}$. $\#\mathbf{Z} = \#\mathbf{N}$ so $\#(\mathbf{Z} \times \mathbf{N}) = \#(\mathbf{N} \times \mathbf{N})$. $\#(\mathbf{N} \times \mathbf{N}) = \mathbf{N}$. So $\#\mathbf{N} \geq \#\mathbf{Q}$. But $\mathbf{N} \subset \mathbf{Q}$, so $\#\mathbf{N} \leq \#\mathbf{Q}$. Therefore $\#\mathbf{N} = \#\mathbf{Q}$, and $\mathbf{Q}$ is countable.
It's possible to construct a bijection from $\mathbf{N}$ to $\mathbf{Q}$ to give a direct proof, but it's very unpleasant.

Useful properties of countable sets

- Any subset of a countable set is either countable or finite.
- Any countable or finite union of countable or finite sets is either countable or finite.
- The Cartesian product of finitely many countable sets is countable.

You can also use these to give a proof that $\mathbf{Q}$ is countable.
Not all sets are countable though. If P is the set of subsets of A then $\#P \neq \#A$, so the set of subsets of $\mathbf{N}$ is not countable.