MAU 22200 Week 2 Lecture 2

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Miscellaneous Announcements

- I emailed the first set of exercises on Monday. If you didn't get an email let me know.
- Blackboard takes hours to process uploaded videos. If a video isn't there, try the module webpage.

More about countable additivity

Countable additivity implies finite additivity, because we can always take a finite collection of sets and add countably many empty sets.

$$
\begin{aligned}
m(E_1 \cup \cdots \cup E_n) &= m(E_1 \cup \cdots \cup E_n \cup \emptyset \cup \emptyset \cup \cdots) \\
&= m(E_1) + \cdots + m(E_n) + m(\emptyset) + m(\emptyset) + \cdots \\
&= m(E_1) + \cdots + m(E_n) + 0 + 0 + \cdots \\
&= m(E_1) + \cdots + m(E_n)
\end{aligned}
$$

Finite additivity doesn't imply countable additivity. Jordan measure is finitely additive, but not countably additive. We'll see other examples later.
Can we find an extension of Jordan measure which is countably additive? Yes, Lebesgue measure is such an extension. Section 1.2 is all about constructing it and proving its properties, including countable additivity.

Can we go beyond countable additivity?

Section 0.0 explained how to define sums even when the number of summands is *uncountable*. So if E_α is a collection of disjoint subsets indexed by $\alpha \in A$ then $\bigcup_{\alpha \in A} E_\alpha$ and $\sum_{\alpha \in A} m(E_\alpha)$ are both well defined, even if A isn't countable. Is
$m\left(\bigcup_{\alpha \in A} E_\alpha\right) = \sum_{\alpha \in A} m(E_\alpha)$?
No! There is no extension of Jordan measure for which this is true. Let $A = [0,1]^d$ and $E_\alpha = \{\alpha\}$ for all $\alpha \in A$. $m(\{\alpha\}) = 0$, so $\sum_{\alpha \in A} m(E_\alpha) = 0$. But $\bigcup_{\alpha \in A} E_\alpha = A$ and $m(A) = 1$. $0 \neq 1$, so there's no way to extend countable additivity to arbitrary additivity.
We can have countable additivity, but we can't have additivity over collections of cardinality equal to or greater than $[0,1]^d$. Is there anything in between? This is undecidable. It can neither be proved nor disproved from the usual axioms of set theory!

Construction of Jordan measure

Inner Jordan measure is defined for all sets and outer Jordan measure is defined for all bounded sets. The inner measure of E is the supremum of the elementary measure of the elementary subsets of A while the outer measure of A is the infimum of the elementary measure of the elementary supersets of A.
Neither inner nor outer Jordan measure has nice properties by itself. Bounded sets are called Jordan measurable if the inner and outer measure are the same, and the Jordan measure is their common value.
Jordan measure does have nice properties. This is connected to problem solving strategy 2.1.1 (Split up equalities into inequalities). You prove equalities for Jordan measure by proving inequalities for inner and outer measure.

Connection to Darboux/Riemann integrals

This is reminiscent of the Darboux theory of integration. That's not an accident!
You can define Darboux upper and lower integrals for arbitrary bounded functions, but neither has nice properties. Integrable functions are those where the upper and lower integrals are the same, and the integral of such a functions is their common value. Integrals do have have nice properties, e.g.

$$\overline{\int_a^b} (f(x) + g(x))\, dx \leq \overline{\int_a^b} f(x)\, dx + \overline{\int_a^b} g(x)\, dx$$

$$\underline{\int_a^b} (f(x) + g(x))\, dx \geq \underline{\int_a^b} f(x)\, dx + \underline{\int_a^b} g(x)\, dx$$

$$\Rightarrow \quad \int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Construction of Lebesgue measure

It's possible to construct Lebesgue measure similarly. One can define Lebesgue outer measure and Lebesgue inner measure. Lebesgue measurable sets ought to be those whose inner and outer measure are the same, and the Lebesgue measure ought to be their common value. Lebesgue outer measure is defined like Jordan outer measure, except you allow countably many disjoint boxes rather than finitely many.

Tao doesn't do this. The definition of inner measure isn't what you might expect, and unbounded sets create technical difficulties. You'll see what he does instead in Subsection 1.2.2. It's inspired by the approach above, but the details are changed to make things work more smoothly.