MAU 22200 Week 2 Lecture 1

John Stalker

Trinity College Dublin

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Q&A sessions

- No one has indicated a scheduling conflict with Q&A's on Fridays, so I'll stick to that unless there's a problem.
- Unless there's a problem, I'll probably stick to Microsoft Teams.
- The recording appears there automatically afterwards.
- You can email questions in advance. That may be useful if there are equations or inequalities which are too complicated to say out loud.
- You don't have to though.
- For exercises which are currently assigned I am more likely to give hints than full solutions.

Exercises

- I'll email the first set of group assignments later today.
- As I said, if you don't want to use your @tcd.ie email for this, let me know.
- Note that in your proofs you are allowed to use the statement of any exercise which occurs before yours in the book.
- The exercises are often designed as a sequence, so it's very likely that you'll want to do that.
- You can, of course, use any earlier theorems, propositions, lemmas, etc.
- You can also use any later ones, *provided this doesn't lead to a circular argument.*

Reading for this week

- Subsection 1.1.2 Jordan Measure
- Subsection 1.1.3 Connection with the Riemann integral
- Introduction to Section 1.2 Lebesgue measure
- Subsection 1.2.1 Properties of Lebesgue outer measure

Elementary measure

Elementary measure applies to elementary sets, i.e. finite unions of boxes. It's non-negative, monotone, finitely additive and assigns the measure 0 to the empty set. Some of those properties are consequences of others. It's translation invariant, but not rotation invariant, because rotating a box doesn't generally produce another box, or finite union of boxes. It has an alternate characterisation as

$$m(E) = \lim_{N \to \infty} \frac{1}{N^d} \# \left(E \cap \frac{1}{N} \mathbf{Z}^d \right).$$

Why isn't this used as the definition? Some important properties are hard to derive from this, e.g. translation invariance. Also, it generalises very badly.
Elementary measure isn't of independent interest, it's only a building block for better theories.

Jordan measure

Jordan measure, by contrast, is actually useful. Again, it's only defined for certain sets, called Jordan measurable sets. That's unavoidable for any well behaved notion of area/volume. It extends elementary measure, in the sense that any elementary set is Jordan measurable and its Jordan measure is its elementary measure.

Jordan measure is non-negative, monotone, finitely additive and assigns the measure 0 to the empty set. It's translation *and* rotation invariant. For $d = 2$ the measure of a ball of radius $r > 0$ is πr^2, while for $d = 3$ the measure is $\frac{4}{3}\pi r^3$. Jordan measure satisfies the axioms of area and volume from last week, except that not every subset of a Jordan measurable set is Jordan measurable.

Jordan measure is closely linked to the Riemann integral.

Problems

So far, “measure”, without any adjective in front, is an undefined term. It will be defined in Definition 1.4.27. Elementary measure and Jordan measure are not measures by that definition!
The general definition requires not just finite additivity but countable additivity. Countable unions of elementary sets needn’t be elementary. Countable unions of Jordan measurable sets needn’t be Jordan measurable.

A geometry problem

What is the volume of the set

$$\{(x,y,z) \in \mathbf{R}^3 \colon x^2z^2 + y^2z^2 + x^2 + y^2 - 1 \le 0\}?$$

In cylindrical coordinates this is $(z^2+1)r^2 \le 1$ so its volume should be

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{1/\sqrt{1+z^2}} r\,dr\,d\theta\,dz = \int_{-\infty}^{\infty}\int_0^{2\pi} \frac{1}{2(1+z^2)}\,d\theta\,dz$$
$$= \int_{-\infty}^{\infty} \frac{\pi}{1+z^2}\,dz = \pi^2$$

Unfortunately, this set isn't Jordan measurable. This is related to the fact that the outer integral is improper. In fact, no unbounded set is Jordan measurable. If we had countable additivity we could slice this set into countably many pieces, e.g. $n - \frac{1}{2} \le z < n + \frac{1}{2}$ and add, to get the expected answer.