MAU 22200 Week 1 Lecture 1

John Stalker

Trinity College Dublin

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Introduction

Part II of MAU22200 covers measure and integration. I'll explain more of what that means later.
Webpage: `http://www.maths.tcd.ie/~stalker/22200`.
I'll try to post (nearly) everything to the webpage and to Blackboard, but may sometimes forget one or the other. If you need to contact me, my email is `stalker@maths.tcd.ie`.
There is a textbook for this part of the module: *An introduction to measure theory* by Terence Tao. You can buy it from the American Mathematical Society for $36, or download it (legally) for free. I'm not going to repeat proofs from the text in lecture, so you do need to acquire a copy and read it. Also, I won't use all the lecture slots for lectures. I'll do one live Q&A session per week. Let me know if there's a lecture slot for which you're not available.

The book

- Measure theory and integration can be confusing, but the book does a good job of presenting it clearly.
- It has a large number of exercises.
- It has useful advice about how to construct proofs.
- It is a bit too advanced for this module. At US universities measure and integration is usually taught later than it is here. The book doesn't really assume anything you don't know though.
- We're not going to cover every section, and even for the the sections we do cover, we won't necessarily cover every subsection.
- Many of the proofs depend on the exercises.

Exercises

The exercises aren't always easy. Ideally each of you would do every exercise as you encounter it. That's not realistic though.

- Read and understand the statement of each exercise.
- Think about how you would try to prove it. The more of an effort you make, the better.
- Each week I'll assign you randomly to a group of two or three and assign that group one problem do to properly.
- Write up a solution collectively and post it to the Blackboard discussion board.
- Read through all the other groups' solutions for the previous week and ask about anything you're unsure of.
- Answer any questions about your group's solution.

What is measure theory?

Measure theory attempts to unify a number of closely related concepts.

Examples include cardinality of sets, probabilities of events, areas of subsets of the plane, volumes of subsets of three dimensional space, etc. In this module we're primarily interested in the last two examples, and their generalisation to n dimensions, but it's helpful to keep the others in mind.

Cardinality The cardinality of a set is non-negative, i.e. $\#A \geq 0$. Cardinality is monotone, i.e. if $A \subset B$ then $\#A \leq \#B$. The cardinality of the empty set is 0, i.e. $\#\emptyset = 0$. Cardinality is additive, i.e. $\#(A \cup B) + \#(A \cap B) = \#A + \#B$. In particular, if $A \cap B = \emptyset$ then $\#(A \cup B) = \#A + \#B$.

Probability Suppose Ω is the set of possible outcomes of an experiment and $\mathbf{P}(A)$ is the probability that the outcome belongs to some subset $A \subset \Omega$. Probability is non-negative, i.e. $\mathbf{P}(A) \geq 0$.

What is measure theory? (Continued)

Probability is monotone, i.e. if $A \subset B$ then $\mathbf{P}(A) \leq \mathbf{P}(B)$. The probability of an outcome in the empty set is 0, i.e. $\mathbf{P}(\emptyset) = 0$. Probability is additive, i.e. $\mathbf{P}(A \cup B) + \mathbf{P}(A \cap B) = \mathbf{P}(A) + \mathbf{P}(B)$. In particular, if $A \cap B = \emptyset$ then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$.

Area The area of a subset of the plane is non-negative, i.e. $\alpha(A) \geq 0$. Area is monotone, i.e. if $A \subset B$ then $\alpha(A) \leq \alpha(B)$. The area of the empty set is 0, i.e. $\alpha(\emptyset) = 0$. Area is additive, i.e. $\alpha(A \cup B) + \alpha(A \cap B) = \alpha(A) + \alpha(B)$. In particular, if $A \cap B = \emptyset$ then $\alpha(A \cup B) = \alpha(A) + \alpha(B)$.

Volume, of course is similar.

There are differences, of course. Probabilities are finite, but cardinalities, areas and volumes can be infinite, for example. The empty set is the *only* set with cardinality 0, but it's not the only set with area 0 or volume 0. But there's enough similarity to try to build a general theory.

Why is measure theory hard?

Let's try to build an axiomatic theory of volume. It seems reasonable to suppose that

- Every set in three dimensional space has a non-negative, but possibly infinite, volume.
- Volume is monotone.
- The volume of the empty set is 0.
- Volume is (finitely) additive.
- Congruent sets have the same volume.
- Balls of radius r have volume $\frac{4}{3}\pi r^3$.

We might expect to need more axioms, but this seems like a good start. In fact, we've already assumed too much. These axioms are logically inconsistent!

Banach-Tarski

Theorem (Banach-Tarski): There are sets E_1, E_2, E_3, E_4, E_5 and F_1, F_2, F_3, F_4, F_5 such that

- E_i is congruent to F_i for each i,
- $E_i \cap E_j = \emptyset$ and $F_i \cap F_j = \emptyset$ when $i \neq j$,
- $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ is a ball of radius 1.
- $F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5$ is the union of two balls of radius 1, which do not intersect.

If the axioms above are satisfied then E_i and F_i have the same volume but also the sum of the volumes of the F_i is twice the the sum of the volumes of the E_i. Note that these sums are finite and positive, so we have a contradiction.

How to escape the contradiction?

There are two lessons to be drawn from Banach-Tarski:

- We need to drop at least one of our axioms.
- No everything which is obvious is true.

The axiom we drop is the first one, that every subset of three dimensional space has a non-negative, but possibly infinite, volume. We don't drop non-negativity, we just don't try to assign a volume to *every* subset.
The only way to deal with the second problem is to be very careful. Don't trust your intuition!

Integration

Despite everything I've just said, this module isn't primarily about measure theory at all. It's mostly about integration.

You need measure theory for a proper discussion of integration though. You may have seen integrals introduced informally as the area under the graph. That's not actually how we'll end up defining them, but it will be a consequence of the definition. Area is an example of a measure, specifically of Lebesgue measure.

One advantage of doing things in a more general context is that you can avoid repeating arguments. For example, the same theorems (Fubini-Tonelli) will govern exchanging two sums, exchanging a sum and an integral, or exchanging two integrals.

Reading for this week

Read Section 0.0 (Preface) of the textbook.
Read Section 2.1 (Problem solving strategies). Parts of that won't make sense yet, but the parts that do will make it easier to read Chapter 1.
Read Section 1.1 (Prologue: The problem of measure), up through the end of Subsection 1.1.1.