MAU 22200 Week 12 Lecture 2

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22 April 2021

Exam Structure

- Structurally, the real exam looks a lot like the practice exam that I posted on Monday.
- There are six questions, three in each part, and you're expected to do two from each part.
- Each question is divided into two or three parts.
- Each question is worth 25 points and the points for the subparts are indicated.
- The instructions are the ones you should expect on the real exam.
- The instructions are a bit different from those used in last semester's exams, which were a bit different from the ones the previous semester's.

Instructions (1/3)

1. This is an open-book exam. You are allowed to use your class notes, textbooks and any material which is directly linked from the module's Blackboard page or from the module's webpage, if it has one. You may not use any other resources except where your examiner has specifically indicated in the "Additional instructions" section below. Similarly, you may only use software if its use is specifically permitted in that section. You are not allowed to collaborate, seek help from others, or provide help to others.
2. If you have any questions about the content of this exam, you may seek clarification from the lecturer using the e-mail address provided. You are not allowed to discuss this exam with others. You are not allowed to send exam questions or parts of exam questions to anyone or post them anywhere.

Instructions (2/3)

3. Unless otherwise indicated by the lecturer in the "Additional instructions" section, solutions must be submitted through Blackboard in the appropriate section of the module webpage by the deadline listed above. You must submit a single pdf file for each exam separately and sign the following declaration in each case. It is your responsibility to check that your submission has uploaded correctly in the correct section.

What does this mean in practice?

- You're allowed to use the textbook. Specifically, you're allowed to use any results stated there, even the ones which are exercises. It's best if you cite whatever you're using, either by name or number, e.g. "by the Monotone Convergence Theorem" or "by Theorem 1.4.43" In this case that's less a matter of academic integrity than a matter of making sure I can figure out what you're doing.

Instructions (3/3)

- You can use any results from lecture. I don't intend to test anything which was done in lecture but not in the text, but if you want to use any of that then you can.
- You can use material from the Blackboard discussion board, more or less.
- Unlike the previous two sources, you shouldn't assume that things posted to the discussion board are correct. You can use it as a source of ideas, but not as an authority.
- I will switch the boards to moderated shortly before the exam, but I won't block read access.
- That's pretty much all you're allowed to use. If you're unsure, ask me by email.

The nature of the questions

All questions are to do with $\mathbf{R}^d$ and Lebesgue measure. You may need results which were stated in a more general context, but you'll only need them in $\mathbf{R}^d$.

The questions are mainly meant to test knowledge of the main results, as listed in Monday's lecture, and the ability to apply them. You aren't expected to have any detailed knowledge of their proofs. You also aren't expected to know all the minor results covered in the book. Part of knowing how to apply the results is figuring out which ones you need for a given problem. You generally won't be told.

Expect to do some extra work, of a routine nature, to get your problem to where you can apply one of the main theorems, and then some more to get the answer in the form you need.

Also, expect to be asked for examples with various properties.

There may also be a small number of (parts of) questions which don't fit either of those patterns.

What do I mean by that? (Q5a)

Here is Question 5, Part a:

> *Suppose $E \subset \mathbf{R}^2$ is Lebesgue measurable and has positive measure. Let F be the set of $x \in \mathbf{R}$ such that the set $\{y \in \mathbf{R} \colon (x, y) \in E\}$ is Lebesgue measurable and has positive measure. Prove that F is Lebesgue measurable and has positive measure.*

This tests one of the main theorems of the module, but which one? It's Fubini's Theorem. Or Tonelli's, you can use either. You need to apply Fubini's Theorem to 1_E.
In more detail, E is a measurable set, so 1_E is a measurable function. If $m(E) < \infty$ then 1_E also absolutely integrable. By Fubini the set of x for which $1_E(x, y)$ is not a measurable function of y is null. That's set of x for which $\{y \in \mathbf{R} \colon (x, y) \in E\}$ is not a measurable set.

More Q5a

By Fubini,

$$\begin{aligned} m(E) &= \int_{\mathrm{R}^2} 1_E(x,y)\,dm(x,y) \\ &= \int_{\mathrm{R}} \left(\int_{\mathrm{R}} 1_E(x,y)\,dm(y) \right) dm(x) \\ &= \int_{\mathrm{R}} m\left(\{y \in \mathbf{R} \colon (x,y) \in E\}\right) dm(x). \end{aligned}$$

If F were of measure 0 we'd have $m\left(\{y \in \mathbf{R} \colon (x,y) \in E\}\right) = 0$ for almost all x and so

$$\int_{\mathrm{R}} m\left(\{y \in \mathbf{R} \colon (x,y) \in E\}\right) dm(x) = 0.$$

But $m(E) > 0$ by assumption. So F is not of measure zero; it must be of positive measure.

Still more Q5a

There's a little more work to be done to deal with the possibility that $m(E) = \infty$, but that's the main idea. You could deal with that by using Fubini-Tonelli in place of Fubini, or just Tonelli. Or you could use Fubini and prove a simple lemma that every subset $\mathbf{R}^d$ of positive measure contains a set of finite positive measure.

If you're very familiar with the book you might notice that there's a result, Corollary 1.7.19, which does almost all of the work for you, but I wouldn't expect that.

Most of the (parts of) questions are like this. There's a theorem, one of the main theorems, which does most of the work for you. The main thing is identify which one (or ones) do the job. Once you've identified the theorem you need to figure out what set, function, sequence, etc. to apply it to. From that point it's mostly straightforward.