MAU 22200 Week 12 Lecture 1

John Stalker

Trinity College Dublin

19 April 2021

Miscellaneous remarks

- There is no assignment this week.
- There is no reading assigned, but you might want to look at Section 2.3 on Probablity Theory. It's short (two pages) and gives a different perspective on measure and integration.
- I'll be posting a sample exam later today to the webpage and Blackboard. There are questions only for this semester's material.
- The plan for this week is
  - Today: a quick overview of the semester
  - Thursday: discussion of the exam
  - Friday: Q&A session as usual

Overview of the Semester: Main Topics

Here's a *very* broad overview of what Semester 2 was about:

- Lebesgue Measure and its Elementary Properties (1.2)
- The Lebesgue Integral and its Elementary Properties (1.3)
- Interchanging Limits and Integrals (1.4)
- Interchanging Derivatives and Integrals (1.6)
- Interchanging Integrals and Integrals (1.7)

Overview of the Semester: Main Concepts

You don't necessarily need the definitions of all of these, but you need to understand what they are:

- Jordan measurable set, Borel set, Lebesgue measurable set, null set, properties which hold almost everywhere,
- Jordan "measure", Lebesgue measure, counting measure
- Riemann integrable function, measurable function, absolutely integrable function,
- Riemann integral of a function, Lebesgue integral of a function, integral of a function with respect to a measure
- Pointwise convergence, uniform convergence, locally uniform convergence, convergence in $L^1(\mathbf{R}^d)$,
- Boolean algebra, σ-algebra, finitely additive measure, (countably additive) measure, pre-measure, outer measure, complete measure, σ-finite measure
- absolutely continuous function, Lipschitz function, function of bounded variation
- product measure

Overview of the Semester: Main Theorems

Of the hundreds of theorems, propositions, lemmas, corollaries, exercises, etc. in the book, these are the main ones to remember:

- Density Theorems:
  - Compactly Supported Continuous Functions are Dense in $L(\mathbf{R}^d)$
  - The Density of Simple Functions are Dense in $L(\mathbf{R}^d)$
- Limit Theorems:
  - The Lebesgue Dominated Convergence Theorem
  - The Monotone Convergence Theorem
  - Fatou's Lemma
- Differentiation Theorems:
  - The First and Second Fundamental Theorems of Calculus
  - The Lebesgue Differentiation Theorem
- The Fubini-Tonelli Theorem

Overview of the Semester: Not Quite So Main Theorems, Useful Inequalities

Of the others, the following are worth mentioning:

- Lusin's Theorem (and its Converse)
- Egorov's Theorem
- Lebesgue's Theorem on Riemann Integrability
- The Borel-Cantelli Lemma
- The Hahn-Carathéodory Extension Theorem
- Various Covering Lemmas: Vitali, Besicovitch, etc.

There are also the basic inequalities:

- Markov's Inequality
- The $L^1(\mathbf{R}^d)$ Triangle Inequality
- The (unnamed) inequality $\left|\int_{\mathrm{R}^d} f(x)\,dx\right| \leq \int_{\mathrm{R}^d} |f(x)|\,dx$
- The Hardy-Littlewood Maximal Inequality

Overview of the Semester: Main Techniques

The following strategies from Section 2.1 are worth remembering:

- Splitting up equalities into inequalities
- Giving yourself an ϵ of room
- Replacing uncountable unions or intersections by countable or finite unions or intersections
- Exploiting Zeno's paradox: a single ϵ can be cut up into countably many sub-ϵs, a.k.a the $\epsilon/2^n$ trick.
- Passing to a subsequence

Density and Associated Concepts

Another technique which isn't explicitly listed in Section 2.1: the density argument.

Density is an application to functions of a more general idea: decompose a large and bad object into a good, but possibly large, part and a small, but possibly bad, part.

Example: An absolutely integrable function is the sum of

- a compactly supported continuous function and a function whose $L^1(\mathbf{R}^d)$ norm can be made arbitrarily small, or
- a bounded continuous function and a function whose support can be made to have arbitrarily small measure.

Similarly, every measurable set is the union of

- a closed set and a set whose outer measure can be made arbitrarily small, or
- a Borel set and a set of measure zero.

Overview of the Semester: Main Interesting Examples I

Sets:

- the empty set
- singletons
- intervals and boxes
- the rationals
- the Cantor set
- the non-measurable sets from Subsection 1.2.3

Functions:

- The indicator functions of any of the sets above
- The function $f(p/q) = 1/q$ on the rationals, extended by zero to the irrationals
- The Weierstrass everywhere continuous nowhere differentiable function from Exercise 1.6.28
- The Cantor function from Exercise 1.6.48

Overview of the Semester: Main Interesting Examples II

Sequences of functions:

- The ones used to construct examples above
- The "escape" sequences of Example 1.4.39-41

Measure spaces:

- $\mathbf{R}^d$, the Lebesgue measurable sets, Lebesgue measure
- $\mathbf{R}^d$, the Borel measurable sets, Borel measure
- Either of the above, restricted a measurable subset
- Any set, all subsets, counting measure
- Any set, itself and the empty set, 1 and 0
- Any set, any σ-algebra on it, the measure assigning ∞ to all non-empty subsets

Usually, if you have an untrue statement and want an example to show that it's untrue then one of the above examples, or a combination of them, will work.