MAU 22200 Week 11 Lecture 2

John Stalker

Trinity College Dublin

15 April 2021

Miscellaneous comments

- This week's exercises are the last exercises.
- There will be Q&A sessions both this week and next week though. Next week's one is your opportunity to ask questions about the exam.
- There's an exercise in Section 1.6, dealing with the Cantor function, which is important, but not quite important enough to do in lecture. It's too hard to assign to a group though, so I'll handle it the way I did for Exercises 1.6.1-1.6.4: I'll post a solution to the discussion board. Look for Exercise 1.6.48 there.

Fubini-Tonelli

Fubini's Theorem (Theorem 1.7.21) gives

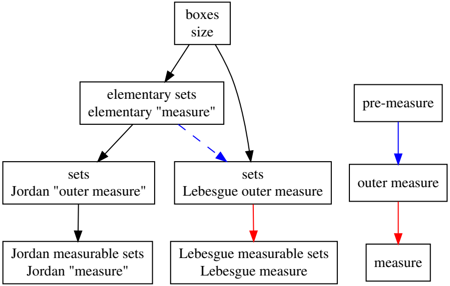
$$\int_{X\times Y} f(x,y)\, d\overline{\mu_X \times \mu_Y}(x,y) = \int_X \left(\int_Y f(x,y)\, d\mu_Y(y) \right) d\mu_X(x)$$
$$= \int_Y \left(\int_X f(x,y)\, d\mu_X(x) \right) d\mu_Y(y)$$

if f is absolutely integrable. We saw that it's not sufficient to assume that $f(x,y)$ is absolutely integrable as a function of y for each x and that $\int_Y f(x,y)\, d\mu_Y(y)$ is absolutely integrable as a function of x, which is annoying, because that's often easier to prove. It *is* sufficient to assume that $f(x,y)$ is absolutely integrable as a function of y for each x and that $\int_Y |f(x,y)|\, d\mu_Y(y)$ is absolutely integrable as a function of x though. That's the Fubini-Tonelli Theorem (Corollary 1.7.23). It's a consequence of Fubini's Theorem and Tonelli's Theorem (Theorem 1.7.18).

Constructing Lebesgue and product measures

It's important not just *that* we can construct product measure but also *how* we construct it.

The left hand side of this diagram summarises how we got Jordan "measure" and Lebesgue measure from boxes in Sections 1.1-2.



The dotted blue arrow reflects the fact that we could just as easily have used elementary sets rather than boxes in the contstruction of Lebesgue outer measure.

The diagram on the right shows the constructions from Section 1.7 which (roughly) parallel those of Section 1.2.

Pre-measure

If X is a set and $\mathcal{B}_0$ is a Boolean algebra on X then a pre-measure is a $\mu_0 \colon \mathcal{B}_0 \to [0, +\infty]$ such that

1. $\mu_0(\emptyset) = 0$,
2. if $E, F \in \mathcal{B}_0$ are disjoint then $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$,
3. if $E_1, E_2, \ldots \in \mathcal{B}_0$ are disjoint and $\bigcup_{n=1}^{\infty} E_n \in \mathcal{B}_0$ then $\mu_0\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu_0(E_n)$.

Some things to note:

- (2) is actually redundant.
- (1) and (2) are the definition of a finitely additive measure.
- If $\mathcal{B}_0$ is a σ-algebra then $E_1, E_2, \ldots \in \mathcal{B}_0$ implies $\bigcup_{n=1}^{\infty} E_n \in \mathcal{B}_0$, so any pre-measure is measure.
- Conversely, every measure is a pre-measure.

The elementary sets aren't a Boolean algebra, but elementary and co-elementary sets together form a Boolean algebra. Elementary "measure" can be extended to a pre-measure.

Outer measure

If X is a set and 2^X is the discrete σ-algebra on X then an outer measure is a $\mu^*\colon 2^X \to [0, +\infty]$ such that

1. $\mu^*(\emptyset) = 0$,
2. if $E, F \in 2^X$ and $E \subset F$ then $\mu^*(E) \leq \mu^*(F)$,
3. if $E_1, E_2, \ldots \in 2^X$ then $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$.

Some things to note:

- Any measure on the discrete σ-algebra is an outer measure, but we rarely have measures on the discrete σ-algebra.
- Outer measures are only rarely even finitely additive.
- It turns out that an outer measure is finitely additive if and only if it's a measure.

Lebesgue outer measure is an outer measure. Jordan outer measure isn't! $E_n = \{q_n\}$, where $q_1, q_2, \ldots$ are the rational elements of $[0, 1]$, violates $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$.

Pre-measure to outer measure

Suppose μ_0 is a pre-measure on the Boolean algebra $\mathcal{B}_0$ on X, μ^* is an outer measure on X, and μ^* extends μ_0, i.e. $\mu^*(E) = \mu_0(E)$ for all $E \in \mathcal{B}_0$. If $E_1, E_2, \ldots \in \mathcal{B}_0$ and $E \subset \bigcup_{n=1}^{\infty} E_n$ then

$$\mu^*(E) \leq \mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n) \leq \sum_{n=1}^{\infty} \mu_0(E_n)$$

This is true for any cover E_1, E_2, $\ldots$ of E by elements of $\mathcal{B}_0$, so $\mu^*(E)$ is less than or equal to the infimum of the right hand side over all such covers.

The reverse inequality doesn't hold in general, but it does in the case of elementary measure and Lebesgue outer measure.

We can turn this into a definition. Given μ_0 we can define μ^* to be the infimum of sums above. This defines an outer measure.

This is more or less how we got Lebesgue outer measure from elementary measure.

Outer measure to measure

The way we got Lebesgue measure from Lebesgue outer measure can't be generalised.
In the Lebesgue case we defined the measure to be the outer measure, but only for a particular collection of sets, defined in terms of open sets. In the general case we can do something similar, but we don't have open sets available. Instead we can use Carathéodory's criterion, that E is measurable if and only if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \backslash E)$ for all A, whether measurable or not.
The proof that this works, i.e. that we get a measure, is fairly complicated.

From pre-measure to measure

You can combine the two constructions, first extending a premeasure μ_0 from a Boolean algebra $\mathcal{B}_0$ on a set X to an outer measure μ^* on 2^X and then restricting μ^* to a measure μ on a σ-algebra $\mathcal{B}$. $\mathcal{B}$ contains $\mathcal{B}_0$ and μ is an extension of μ_0. That's the content of the Hahn-Carathédory Extension Theorem (Theorem 1.7.8). The theorem just says that the pre-measure μ_0 can be extended from $\mathcal{B}_0$ to a measure on a σ-algebra $\mathcal{B}$ containing $\mathcal{B}_0$. If you construct the extension as above then $(X, \mathcal{B}, \mu)$ is a complete measure space. If you construct it in some other way you might not get a complete measure space. Borel and Lebesgue measure are both extensions of Jordan "measure", but only Lebesgue is complete.

Product measure

Given measure spaces $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ let $\mathcal{B}_0$ be the set of subsets of $X \times Y$ of the form $S = E_1 \times F_1 \cup \cdots \cup E_k \times F_k$ where $E_1, \ldots, E_k \in \mathcal{B}_X$, $F_1, \ldots, F_k \in \mathcal{B}_Y$, and $E_1 \times F_1, \ldots, E_k \times F_k \in \mathcal{B}_Y$, are disjoint. Then

- $\mathcal{B}_0$ is a Boolean algebra.
- $\mu_0(S) = \mu_X(E_1)\mu_Y(F_1) + \cdots + \mu_X(E_k)\mu_Y(F_k)$ is well defined, i.e. independent on how we decompose S.
- μ_0 is a pre-measure.
- By Hahn-Carathéodory there are a σ-algebra $\mathcal{B}$ and a measure μ on $X \times Y$ such that
  - $E \times F \in \mathcal{B}$ whenever $E \in \mathcal{B}_X$ and $F \in \mathcal{B}_Y$
  - $\mu(E \times F) = \mu_X(E)\mu_Y(F)$.
- If $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ are σ-finite then there's only one such extension.

This is Proposition 1.7.11.

Using Hahn-Carathéodory like this is the most common way to construct measures.