MAU 22200 Week 11 Lecture 1

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Miscellaneous remarks

- It looks like the mismatch in the numbering extends into Section 1.6.
- There are at least three versions of the book.
  - `gsm126.pdf`, the version published by the AMS, which has "Purchased from American Mathematical Society for the exclusive use of ..." at the bottom of each page.
  - `gsm-126-tao5-measure-book.pdf`, linked from Terry's page, linked from the module webpage, which has "Author's preliminary version made available with permission of the publisher, the American Mathematical Society" at the bottom of each page.
  - `measure-book1.pdf`, linked from a different page of Terry's, which has nothing at the bottom of each page.
- I'm using the first of these. If you're using a different version, subtract 1 from exercise numbers for this week's problems in Section 1.6. Leave numbers in Section 1.7 unchanged.

Reading

For this week:

- Finish reading Section 1.6 if you haven't already.
- Read the introduction to Section 1.7 (Outer measures, pre-measures, product measures).
- Read Subsection 1.7.1 (Outer measures and the Carathéodory extension theorem)
- Read Subsection 1.7.2 (Pre-measures).
- Read Subsection 1.7.4 (Product measure).

You don't have to read Subsections 1.7.3 (Lebesgue-Stieltjes measure).

Fubini's Theorem (Continuous case)

If $f : [a, b] \times [c, d] \to \mathbf{C}$ is continuous then

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_{[a,b]\times[c,d]} f(x, y) = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

All integrands are continuous, hence Riemann integrable, hence absolutely integrable in the sense of Lebesgue. It therefore doesn't matter whether we interpret these as Riemann integrals or Lebesgue integrals.
We've used one obvious fact (the restriction of a continuous function is continuous) and one less obvious fact (the inner integrals depend continously on the remaining variable).

Example

$$\int_0^1 \int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\, dy\, dx = \int_0^1 \int_0^1 \frac{\partial}{\partial y}\left(\frac{1}{2}\frac{xy^2}{(x^2+y^2)^2}\right)\, dy\, dx$$
$$= \int_0^1 \left(\frac{1}{2}\frac{x}{(1+x^2)^2}\right)\, dx$$
$$= \int_0^1 \frac{\partial}{\partial x}\left(-\frac{1}{8}\frac{1-x^2}{1+x^2}\right)\, dy = -\frac{1}{8}.$$

$$\int_0^1 \int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\, dx\, dy = \int_0^1 \int_0^1 \frac{\partial}{\partial x}\left(-\frac{1}{2}\frac{x^2y}{(x^2+y^2)^2}\right)\, dx\, dy$$
$$= \int_0^1 \left(-\frac{1}{2}\frac{y}{(1+y^2)^2}\right)\, dy$$
$$= \int_0^1 \frac{\partial}{\partial y}\left(\frac{1}{8}\frac{1-y^2}{1+y^2}\right)\, dy = \frac{1}{8}.$$

What went wrong?

Unfortunately $-\frac{1}{8} \neq \frac{1}{8}$. What went wrong?
The theorem is correct as stated.
The derivative calculations and the uses of the Fundamental Theorem of Calculus are fine.
The function

$$f(x, y) = \frac{xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

is undefined at $(0, 0)$. It's defined and continuous everywhere else. We can extend it to $(0, 0)$ by defining $f(0, 0) = 0$. For each x then $f(x, y)$ is a continuous function of y, and vice versa. It's not a continuous function of (x, y) though.
The word "continuous" in the statement of the theorem means "continuous", not "continuous almost everywhere" or "continuous in each variable separately" or anything else!

Fubini for absolutely integrable functions?

Is the following true?
If $E \subset \mathbf{R}^d$ and $F \subset \mathbf{R}^{d'}$ are measurable sets then $E \times F \subset \mathbf{R}^{d+d'}$ is measurable. If $f : E \times F \to \mathbf{C}$ is absolutely integrable then

$$\int_E \int_F f(x,y)\, dm^{d'}(y)\, dm^d(x) = \int_{E\times F} f(x,y)\, dm^{d+d'}(x,y)$$
$$= \int_F \int_E f(x,y)\, dm^d(x)\, dm^{d'}(y)$$

I've replaced

- the intervals $[a,b]$ and $[c,d]$ in $\mathbf{R}$ with measurable sets E and F in $\mathbf{R}^d$ and $\mathbf{R}^{d'}$,
- Riemann integrals in $\mathbf{R}$ and $\mathbf{R}^2$ with Lebesgue integrals in $\mathbf{R}^d$, $\mathbf{R}^{d'}$ and $\mathbf{R}^{d+d'}$, and
- continuity with absolute integrability.

Is this plausible? (1/3)

Is $E \times F$ measurable? Yes, by Exercise 1.2.22(ii).

Is our calculation with $f(x, y) = \frac{xy(x^2-y^2)}{(x^2+y^2)^3}$ a counterexample? No, f is measurable but it isn't absolutely integrable. It does mean we need absolute integrability in (x, y) though, not in x and y separately.

Do all the integrals make sense, assuming f is absolutely integrable?

$$\int_{E \times F} f(x, y)\, dm^{d+d'}(x, y)$$

does, more or less by definition.

$$\int_{F} f(x, y)\, dm^{d'}(y)$$

does if $f(x, y)$ is an absolutely integrable function of y for fixed values of x.

Is this plausible? (2/3)

Unfortunately $f(x, y)$ doesn't have to be an absolutely integrable function of y for fixed values of x. It doesn't even have to be measurable.

Choose a null set $N \subset E$ and an unmeasurable set $U \subset F$. Let $P = N \times U$ and $f = 1_P$. P is a null set. Indeed, P is subset of $N \times F$ and $m^{d+d'}(N \times F) = m^d(N)m^{d'}(F) = 0m^{d'}(F) = 0$, by Exercise 1.2.22(ii), so $N \times F$ is a null set. Lebesgue measure is complete, i.e. subsets of null sets are null. The indicator function of a set is absolutely integrable if and only if the set is measurable and of finite measure, so 1_P is absolutely integrable.

For $x \in N$ we have $f(x, y) = 1_U(y)$. 1_U is not a measurable function because U is not a measurable set.

So $\int_F 1_P(x, y)\, dm^{d'}(y)$ is undefined for $x \in N$. It's still defined almost everywhere though, so we may be okay.

Is this plausible? (3/3)

At best we can expect

$$\int_F f(x, y)\, dm^{d'}(y)$$

to be defined almost everywhere. We still need it to be an absolutely integrable function of x in order to make sense of

$$\int_E \int_F f(x, y)\, dm^{d'}(y)\, dm^d(x).$$

Is it? As it turns out, yes!
Similar remarks apply to the inner and outer integrals in

$$\int_F \int_E f(x, y)\, dm^d(x)\, dm^{d'}(y).$$

Generalisation

If we allow the inner integrals in repeated integrals to be defined almost everywhere, the absolutely integrable Fubini makes sense and is true. That's actually Fubini's Theorem. What I called "Fubini's Theorem" for continuous functions is much older. Terry's "Fubini's Theorem" is more modern.
Just like the convergence theorems make sense for general measure spaces, so does Fubini's Theorem. Only two properties of Lebesgue measure are needed:

- Completeness: Every subset of a null set is null.
- σ-finiteness: There is a countable collection of sets of finite measure whose union is everything.

We need a notion of product measures to state the generalised Fubini Theorem. Those are useful anyway, particularly for Probability Theory. It's even possible to take products of arbitrary collections of measure spaces, not just two. That's the point of Section 2.4 of the book.

Fubini for absolutely integrable functions (Thm 1.7.21)

Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be complete σ-finite measure spaces, and let $f : X \times Y \to \mathbf{C}$ be absolutely integrable with respect to $\overline{\mu_X \times \mu_Y}$. Then:

- For μ_X-almost every $x \in X$, the function $y \mapsto f(x, y)$ is absolutely integrable with respect to μ_Y and, in particular, $\int_Y f(x, y)\, d\mu_Y(y)$ exists. Furthermore, the (μ_X-almost everywhere defined) map $x \mapsto \int_Y f(x, y)\, d\mu_Y(y)$ is absolutely integrable with respect to μ_X.
- For μ_Y-almost every $y \in Y$, the function $x \mapsto f(x, y)$ is absolutely integrable with respect to μ_X and, in particular, $\int_X f(x, y)\, d\mu_X(x)$ exists. Furthermore, the (μ_Y-almost everywhere defined) map $y \mapsto \int_X f(x, y)\, d\mu_X(x)$ is absolutely integrable with respect to μ_Y.
- We have

$$\int_{X \times Y} f(x, y)\, d\overline{\mu_X \times \mu_Y}(x, y) = \int_X \left(\int_Y f(x, y)\, d\mu_Y(y) \right) d\mu_X(x)$$
$$= \int_Y \left(\int_X f(x, y)\, d\mu_X(x) \right) d\mu_Y(y).$$