MAU 22200 Week 10 Lecture 1

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Miscellaneous Remarks

- There a four exercises in the introduction to Section 1.6, the part before Subsection 1.6.1 begins. Rather than assign them to groups I did them and have posted solutions to the discussion board.
- There's a converse to Lusin's Theorem buried in the solution to Exercise 1.6.1(iii), which is of independent interest. More on this in a bit.
- Section 1.6 is less linear than previous ones. There are often exercises between when a theorem is stated and proved, so you need to be more careful to avoid circularity.
- Today I mostly want to point out three useful results from Section 1.6: Continuity of Translation in L^1, the Hardy-Littlewood Maximal Inequality and the Lebesgue Differentiation Theorem.

Lebesgue's Theorem on Riemann Integrability and Lusin's Theorem on Lebesgue Measurability

Compare the following:

- Lebesgue's Theorem: $f : \mathbf{R} \to \mathbf{R}$ is Riemann integrable on closed intervals if and only if it is bounded and is continuous almost everywhere. I proved the "if" part in Week 5 Lecture 1. The "only if" part is also true. "f is continuous almost everywhere" can be replaced by "for every $\epsilon > 0$ there is an E with $m(E) < \epsilon$ such that f is continuous at all points of $\mathbf{R} \backslash E$". There's also a generalisation to integrals over Jordan measurable sets in $\mathbf{R}^d$.
- Lusin's Theorem and its converse: $f : \mathbf{R}^d \to \mathbf{R}$ is measurable if and only if for all $\epsilon > 0$ there is an E with $m(E) < \epsilon$ such that the restriction of f to $\mathbf{R}^d \backslash E$ is continuous. The "if" part is proved in the solution I posted to Exercise 1.6.1(iii). The "only if" part is Exercise 1.3.23, a generalisation of Theorem 1.3.28 (Lusin's Theorem).

Continuity of Translation in $L^1(\mathbf{R}^d)$

Section 1.6 is mostly about versions of the Fundamental Theorem of Calculus, but there are other useful facts in there.
One is Proposition 1.6.13: If f is absolutely integrable then

$$\lim_{h \to 0} \int_{\mathbf{R}^d} |f(x-h) - f(x)|\, dx = 0.$$

This would be easy if we could exchange the limit and integral, but Lebesgue Dominated Convergence doesn't apply here.
Instead it's proved by a density argument. More precisely it's proved using the fact that the compactly supported continuous functions are a dense subset of $L^1/(\mathbf{R}^d)$, i.e. that any function in $L^1(\mathbf{R}^d)$ can be approximated arbitrarily well by compactly supported continuous functions (Theorem 1.3.20).

Markov and Hardy-Littlewood Inequalities

Another useful result in Section 1.6 is the Hardy-Littlewood Maximal Inequality.

Suppose $h\colon \mathbf{R}^d \to \mathbf{C}$ is absolutely integrable.

- By Markov's Inequality (Lemma 1.3.15),

$$m\left(\left\{x \in \mathbf{R}^d : |h(x)| \geq \lambda\right\}\right) \leq \frac{1}{\lambda} \int_{\mathrm{R}^d} |h(y)|\, dy.$$

- By the Hardy-Littlewood Maximal Inequality (Theorem 1.6.20),

$$m\left(\left\{x \in \mathbf{R}^d : Mh(x) \geq \lambda\right\}\right) \leq \frac{C_d}{\lambda} \int_{\mathrm{R}^d} |h(y)|\, dy,$$

where

$$Mh(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |h(y)|\, dy.$$

The Lebesgue Differentiation Theorem

Yet another is Theorem 1.6.19 (the Lebesgue Differentiation Theorem): If f is absolutely integrable then, almost everywhere,

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy = f(x)$$

In other words, the averages of an absolutely integrable function over balls of radius r tend to the function as almost everywhere. You can't replace "almost everywhere" by "everywhere", as the counterexample $f = 1_{\mathbb{Q}^d}$ shows.
Despite the name, there is no differentiation involved. For $d = 1$ we can rewrite the equation as

$$\lim_{r \to 0} \frac{F(x+r) - F(x-r)}{2r} = f(x),$$

i.e. a limit of "difference quotients" for $F(y) = \int_{(-\infty,y)} f(x)\, dx$.

Exceptional Sets

Theorem 1.3.20 says that compactly supported continuous functions are dense in $L^1(\mathbf{R}^d)$, so there is a g_n such that

$$\int_{\mathrm{R}^d} |f(y) - g_n(y)|\, dy < \frac{1}{2^{2n}}.$$

By the Markov and Hardy-Littlewood inequalities, with $h = f - g$ and $\lambda = \frac{1}{2^n}$, we have $m(A_n) \leq \frac{1}{2^n}$ and $m(B_n) \leq \frac{C_d}{2^n}$, where

$$A_n = \left\{x \in \mathbf{R}^d : |f(x) - g_n(x)| \geq \frac{1}{2^n}\right\},$$

$$B_n = \left\{x \in \mathbf{R}^d : \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g_n(y)|\, dy \geq \frac{1}{2^n}\right\}.$$

Let $D_n = A_n \cup B_n$. Then $m(D_n) \leq \frac{C_d+1}{2^n}$ and for $x \in \mathbf{R}^d \backslash D_n$ we have $|f(x) - g_n(x)| < \frac{1}{2^n}$ and $\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g_n(y)|\, dy < \frac{1}{2^n}$.

The Three Basic Inequalities

g_n is compactly supported continuous, so it's uniformly continuous. There is therefore a $\delta_n > 0$ such that if $|x - y| < \delta_n$ then $|g_n(x) - g_n(y)| < \frac{1}{2^n}$. If $r < \delta_n$ then $|g_n(x) - g_n(y)| < \frac{1}{2^n}$ for all $y \in B(x, r)$. The same bound holds for its average value:

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |g_n(x) - g_n(y)| \, dy < \frac{1}{2^n}.$$

If $x \in \mathbf{R}^d \backslash D_n$ then

$$|f(x) - g_n(x)| = \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - g_n(x)| \, dy < \frac{1}{2^n}$$

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g_n(y)| \, dy < \frac{1}{2^n}$$

Now apply the triangle inequality
$|f(y) - f(x)| \leq |f(y) - g_n(y)| + |g_n(y) - g_n(x)| + |g_n(x) - f(x)|$.

More Exceptional Sets

We now have a $\delta_n > 0$ and a set D_n with $m(D_n) \leq \frac{C_d+1}{2^n}$ such that if $r < \delta_n$ and $x \in \mathbf{R}^d \backslash D_n$ then

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy < \frac{3}{2^n}.$$

Let $E_m = \bigcup_{n>m} D_n$. Then $m(E_m) \leq \sum_{n>m} m(D_n) < \frac{C_d+1}{2^m}$. If $x \in \mathbf{R}^d \backslash E_m$ then $x \in \mathbf{R}^d \backslash D_n$ for all $n > m$. So, for all $n > m$ there is a $\delta_n > 0$ such that if $r < \delta_n$ then

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy < \frac{3}{2^n}.$$

So for $x \in \mathbf{R}^d \backslash E_m$ we have

$$\lim_{r \to 0} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0.$$

Proof of Lebesgue Differentiation Theorem (Conclusion)

Let E be the set of x such that

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy \neq 0,$$

because the limit either doesn't exist or isn't 0. We just saw that $E \subset E_m$. $m(E_m) < \frac{C_d + 1}{2^m}$. By monotonicity $m(E) < \frac{C_d + 1}{2^m}$. This is true for all m, so $m(E) = 0$. In other words,

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0.$$

almost everywhere.

$$\left| \int_{B(x,r)} |f(y) - f(x) \, dy \right| \leq \int_{B(x,r)} |f(y) - f(x)| \, dy$$

so, almost everywhere,

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, dy = \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x) \, dy = f(x).$$