MAU 34804 Lecture 28

John Stalker

Trinity College Dublin

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Main theorem on exchange economies

Theorem 8.11: Suppose that $\overline{x}_{hi} > 0$ for $1 \leq h \leq m$ and $1 \leq i \leq n$ and that $u_h \colon \mathbf{R}_+^n \to \mathbf{R}$ is continuous, strictly increasing and quasiconcave for $1 \leq h \leq m$. Then there are positive p_i^* for $1 \leq i \leq n$ and non-negative x_{hi}^* for $1 \leq h \leq m$ and $1 \leq i \leq n$ such that

$$\sum_{i=1}^{n} p_i^* x_{hi}^* = \sum_{i=1}^{n} p_i^* \overline{x}_{hi}$$

for $1 \leq h \leq m$ and

$$\sum_{h=1}^{m} x_{hi}^* = \sum_{h=1}^{m} \overline{x}_{hi}$$

for $1 \leq i \leq n$. Also, for any $1 \leq h \leq m$ and any $(x_1, \ldots, x_n) \in \mathbf{R}_+^n$,

$$\sum_{i=1}^{n} p_i^* x_i \leq \sum_{i=1}^{n} p_i^* \overline{x}_{hi} \Rightarrow u_h(x_1, \ldots x_n) \leq u_h\left(x_{h1}^*, \ldots, x_{hn}^*\right).$$

Economic interpretation

- $\overline{x}_{hi}$ is the amount of good i initially held by household h.
- u_h is the utility function for household h.
- p_i^* is the market clearing price of good i.
- x_{hi}^* is the amount of good i held by household h after redistribution.
- The equation $\sum_{i=1}^n p_i^* x_{hi}^* = \sum_{i=1}^n p_i^* \overline{x}_{hi}$ expresses the fact that each household breaks even.
- The equation $\sum_{h=1}^m x_{hi}^* = \sum_{h=1}^m \overline{x}_{hi}$ expresses the fact that goods are conserved.
- The implication

$$\sum_{i=1}^n p_i^* x_i \leq \sum_{i=1}^n p_i^* \overline{x}_{hi} \Rightarrow u_h(x_1, \ldots x_n) \leq u_h\left(x_{h1}^*, \ldots, x_{hn}^*\right)$$

  expresses the fact that each household's utility is maximised, subject to its budget constraint.

Proof of Theorem 8.11

Proof of Theorem 8.11: Let $\bar{\mathbf{x}}_h = (\bar{x}_{h1}, \ldots, \bar{x}_{hn})$, $\mathbf{x}_h^* = (x_{h1}^*, \ldots, x_{hn}^*)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p}^* = (p_1^*, \ldots, p_n^*)$,

$$\Delta = \left\{ \mathbf{p} \in \mathbf{R}_+^n : \sum_{i=1}^{n} p_i = 1 \right\},$$

$$B_{\mathbf{c}}(\mathbf{p}, w) = \left\{ \mathbf{x} \in \mathbf{R}_+^n : \mathbf{x} \leq \mathbf{c}, \mathbf{p} \cdot \mathbf{x} \leq w \right\},$$

$$\hat{V}_{\mathbf{c},h}(\mathbf{p}) = \max_{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, \mathbf{p} \cdot \bar{\mathbf{x}}_h)} u_h(\mathbf{x})$$

$$\hat{\xi}_{\mathbf{c},h}(\mathbf{p}) = \left\{ \mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, \mathbf{p} \cdot \bar{\mathbf{x}}_h) : u_h(\mathbf{x}) = \hat{V}_{\mathbf{c},h}(\mathbf{p}) \right\},$$

$$\mathbf{d}_{\mathbf{c}} = \sum_{h=1}^{m} \hat{\xi}_{\mathbf{c},h}$$

$$\mathbf{s} = \sum_{h=1}^{m} \mathbf{x}_h.$$

Proof of Theorem 8.11 continued

By Proposition 8.9 $\mathbf{d_c}\colon \Delta \rightrightarrows \mathbf{R}^n_+$ is non-empty valued, compact valued, convex valued and upper hemicontinuous if $\mathbf{c} \gg \mathbf{0}$. Also, $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{s}$ if $\mathbf{x} \in \mathbf{d_c}(\mathbf{p})$.
Define $\zeta_\mathbf{c}\colon \Delta \rightrightarrows \mathbf{R}^n$ by

$$\zeta_\mathbf{c}(\mathbf{p}) = \{\mathbf{z} \in \mathbf{R}^n \colon \mathbf{z} + \mathbf{s} \in \mathbf{d_c}(\mathbf{p})\}.$$

Then $\mathbf{p} \cdot \mathbf{z} \leq 0$ for all $\mathbf{z} \in \zeta_\mathbf{c}(\mathbf{p})$.
$\zeta_\mathbf{c}(\Delta) \subseteq \{\mathbf{z} \in \mathbf{R}^n \colon -\mathbf{s} \leq \mathbf{z} \leq \mathbf{c} - \mathbf{s}\}$, which is compact, so by Theorem 8.10 there are $\mathbf{p}^* \in \Delta$ and $\mathbf{z}^* \in \zeta_\mathbf{c}(\mathbf{p}^*)$ such that $\mathbf{z}^* \leq \mathbf{0}$.
Let $\mathbf{y} = \mathbf{z}^* + \mathbf{s}$. Then $\mathbf{y} \in \mathbf{d_c}(\mathbf{p}^*)$ so there are $\mathbf{x}^*_1, \ldots, \mathbf{x}^*_m$ such that $\mathbf{x}^*_h \in \hat{\xi}_{\mathbf{c},h}(\mathbf{p}^*)$ and $\mathbf{y} = \sum_{h=1}^m \mathbf{x}^*_h \leq \mathbf{s}$. $\mathbf{y} \leq \mathbf{s}$ because $\mathbf{z}^* \leq 0$. Also, $\mathbf{x}^*_h \geq \mathbf{0}$ for all h so $\mathbf{x}^*_h \leq \mathbf{s}$. If we choose $\mathbf{c} \gg \mathbf{s}$ then $\mathbf{x}^*_h \ll \mathbf{c}$.
$\mathbf{x}^*_h \in \xi_{\mathbf{c},h}(\mathbf{p}^*)$, i.e. $\mathbf{x}^*_h$ maximises u_h over $B_\mathbf{c}(\mathbf{p}^*, \mathbf{p}^* \cdot \bar{\mathbf{x}}_h)$. Let

$$N = \left\{\mathbf{x} \in \mathbf{R}^n_+ \colon \mathbf{x} \ll \mathbf{c}\right\}.$$

Then N is an open neighbourhood of $\mathbf{x}^*_h$ in $\mathbf{R}^n_+$. $\mathbf{x}^*_h$ maximises u_h over $N \cap B(\mathbf{p}^*, \mathbf{p}^* \cdot \bar{\mathbf{x}}_h)$, so by Proposition 8.4 it maximises u_h over all of $B(\mathbf{p}^*, \mathbf{p}^* \cdot \bar{\mathbf{x}}_h)$. Also, $\mathbf{p}^* \cdot \mathbf{x}^*_h = \mathbf{p}^* \cdot \bar{\mathbf{x}}_h$, hence $\mathbf{p}^* \cdot \mathbf{y} = \mathbf{p}^* \cdot \mathbf{s}$.

Proof of Theorem 8.11 concluded

If $p_i^* = 0$ then we could increase $u_h(\mathbf{x}^*)$ while continuing to satisfy the budget constraint, by increasing x_{hi}^* while leaving x_{hj}^* constant for all $j \neq i$. Since this can't happen, we know that $p_i^* > 0$ for all i. We've already seen that $\mathbf{y} \leq \mathbf{s}$ and $\mathbf{p}^* \cdot \mathbf{y} = \mathbf{p}^* \cdot \mathbf{s}$, so $\mathbf{y} = \mathbf{s}$. In other words,

$$\sum_{h=1}^{m} \mathbf{x}_h^* = \sum_{h=1}^{m} \bar{\mathbf{x}}_h.$$

This completes the proof of Theorem 8.11.

Note that the idea of rationing was useful for technical reasons, but ultimately doesn't matter. None of the rationing constraints are active in the solution.

For a bit more about excess demand, see Sections 8.9 and 8.10.