MAU 34804 Lecture 27

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Addition of compact valued correspondences

If $\Omega \subseteq \mathbf{R}^k$ and $\xi_1, \ldots \xi_m \colon \Omega \rightrightarrows \mathbf{R}^n$ then we define $\sum_{h=1}^m \xi_h \colon \Omega \rightrightarrows \mathbf{R}^n$ by saying that $\left(\sum_{h=1}^m \xi_h\right)(\mathbf{p})$ is

$$\left\{ \mathbf{y} \in \mathbf{R}^n \colon \exists (\mathbf{x}_1, \ldots \mathbf{x}_m) \in \xi_1(\mathbf{p}) \times \cdots \times \xi_m(\mathbf{p}) \colon \sum_{h=1}^m \mathbf{x}_h = \mathbf{y} \right\}.$$

Proposition 8.7 If $\xi_1, \ldots \xi_h$ are non-empty valued, compact valued and upper hemicontinuous then so is $\sum_{h=1}^m \xi_h$.
We will use the following criterion for upper hemicontinuity.
Proposition 2.16: Suppose $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$ and $\Phi \colon X \rightrightarrows Y$ is compact valued. Then Φ is upper hemicontinuous if and only if for every $\mathbf{p} \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that

$$\Phi(B_X(\mathbf{x}, \delta)) \subseteq B_Y(\Phi(\mathbf{p}), \epsilon).$$

Note that B's here are balls rather than budget correspondences.

Proof of Proposition 8.7

Proof of Proposition 8.7: Each $\xi_h(\mathbf{p})$ is compact, so their Cartesian product is compact. Addition of vectors is continuous and the image of a compact set under a continuous function is compact so $\left(\sum_{h=1}^m \xi_h\right)(\mathbf{p})$ is compact.
Let $\epsilon > 0$. Then $\epsilon/m > 0$. Since each ξ_h is upper hemicontinuous there is a $\delta_h > 0$ such that

$$\xi_h(B_\Omega(\mathbf{p}, \delta_h)) \subseteq B_{\mathbf{R}^n}(\xi_h(\mathbf{p}), \epsilon/m)$$

by Proposition 2.16. Let $\delta = \min_{1 \leq h \leq m} \delta_h$. Suppose $\mathbf{q} \in B_\Omega(\mathbf{p}, \delta)$ and $\mathbf{z} \in \left(\sum_{h=1}^m \xi_h\right)(\mathbf{q})$. Then there are $\mathbf{w}_1 \in \xi_1(\mathbf{q})$, ..., $\mathbf{w}_m \in \xi_m(\mathbf{q})$ such that $\sum_{h=1}^m \mathbf{w}_h = \mathbf{z}$. For each h then

$$\mathbf{w}_h \in \xi_h(\mathbf{q}) \subseteq \xi_h(B_\Omega(\mathbf{p}, \delta)) \subseteq \xi_h(B_\Omega(\mathbf{p}, \delta_h)) \subseteq B_{\mathbf{R}^n}(\xi_h(\mathbf{p}), \epsilon/m).$$

In other words there is an $\mathbf{x}_h \in \xi_h(\mathbf{p})$ such that $\|\mathbf{w}_h - \mathbf{x}_h\| < \epsilon/m$.

Proof of Proposition 8.7 concluded

From $\|\mathbf{w}_h - \mathbf{x}_h\| < \epsilon/m$ and the triangle inequality we get $\|\mathbf{z} - \mathbf{y}\| < \epsilon$, where

$$\sum_{h=1}^{m} \mathbf{x}_h = \mathbf{y}, \quad \sum_{h=1}^{m} \mathbf{w}_h = \mathbf{z}.$$

$\mathbf{x}_h \in \xi_h(\mathbf{p})$, so $\mathbf{y} \in \left(\sum_{h=1}^{m} \xi_h\right)(\mathbf{p})$ and hence $\mathbf{z} \in B_{\mathbf{R}^n}\left(\left(\sum_{h=1}^{m} \xi_h\right)(\mathbf{p}), \epsilon\right)$. But $\mathbf{z}$ was an arbitrary element of $\left(\sum_{h=1}^{m} \xi_h\right)(B_\Omega(\mathbf{q}, \delta))$ and $\mathbf{q}$ an arbitrary element of $B_\Omega(\mathbf{p}, \delta)$. In other words, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left(\sum_{h=1}^{m} \xi_h\right)(B_\Omega(\mathbf{p}, \delta)) \subseteq B_{\mathbf{R}^n}\left(\left(\sum_{h=1}^{m} \xi_h\right)(\mathbf{p}), \epsilon\right).$$

But then $\sum_{h=1}^{m} \xi_h$ is upper hemicontinuous, by Proposition 2.16. This completes the proof of Proposition 8.7.

Proposition 8.8

If household h has an initial endowment of $\overline{\mathbf{x}}_h$ and the prevailing prices are $\mathbf{p}$ then they can sell that endowment for a wealth of $\mathbf{p} \cdot \overline{\mathbf{x}}_h$ and buy whatever bundle of goods maximises their utility for that price/wealth pair.

Proposition 8.8: Suppose the $\overline{\mathbf{x}}_h \gg \mathbf{0}$, $\mathbf{c} \gg \mathbf{0}$. Suppose further that $u_h \colon \mathbf{R}^n_+ \to \mathbf{R}$ is quasiconcave, strictly increasing and continuous. Let $\Delta \subseteq \mathbf{R}^n_+$ and $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbf{R}^n_+$ be defined by

$$\Delta = \left\{ \mathbf{p} \in \mathbf{R}^n_+ \colon \sum_{j=1}^{n} p_i = 1 \right\},$$

$$\hat{V}_{\mathbf{c},h}(\mathbf{p}) = \max_{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, \mathbf{p} \cdot \overline{\mathbf{x}}_h)} u_h(\mathbf{x}),$$

$$\hat{\xi}_{\mathbf{c},h}(\mathbf{p}) = \left\{ \mathbf{x} \in B_{\mathbf{c}}\left(\mathbf{p}, \mathbf{p} \cdot \overline{\mathbf{x}}_h\right) : u_h(\mathbf{x}) = \hat{V}_{\mathbf{c},h}(\mathbf{p}) \right\}$$

Then $\hat{\xi}_{\mathbf{c},h}$ is non-empty valued, compact valued, convex valued and upper hemicontinuous.

Comments on Proposition 8.8

- We've assumed that $\bar{\mathbf{x}}_h \gg \mathbf{0}$. That was purely to ensure that the initial wealth of the household is not zero, even if some goods are free. This assumption can be weakened.
- We've also assumed rationing with quantity limits $\mathbf{c} \gg \mathbf{0}$. This assumption prevents us from "buying" unlimited quantities of free goods.
- The assumption that $\sum_{i=1}^{n} p_i = 1$ looks unnatural, and is. It's harmless though, as long as there is at least one non-free good, since the quantities bought and sold by a utility maximising household are unchanged if all prices are multiplied by a positive constant. It's technically useful, because it restricts prices to a compact set.

Proof of Proposition 8.8

Proof of Proposition 8.8: Let $\psi_h \colon \Delta \to \hat{\Gamma}^n$ be defined by

$$\psi_h(\mathbf{p}) = (\mathbf{p}, \mathbf{p} \cdot \bar{\mathbf{x}}_h).$$

ψ_h is continuous. By Proposition 8.6 $V_{\mathbf{c}} \colon \hat{\Gamma}^n \to \mathbf{R}$ is continuous and $\xi_{\mathbf{c}} \colon \hat{\Gamma}^n \rightrightarrows \mathbf{R}^n_+$ is non-empty valued, compact valued, convex valued and upper hemicontinuous. So the same holds for

$$\hat{V}_{\mathbf{c},h} = V_{\mathbf{c}} \circ \psi_h$$

and

$$\hat{\xi}_{\mathbf{c},h} = \xi_{\mathbf{c}} \circ \psi_h.$$

This completes the proof of Proposition 8.8.

Corollary 8.9

Corollary 8.9: With notation and hypotheses as in Proposition 8.8, $\mathbf{d}_c \colon \Delta \rightrightarrows \mathbf{R}^n_+$, defined by

$$\mathbf{d}_c = \sum_{h=1}^{m} \hat{\xi}_{\mathbf{c},h},$$

is non-empty valued, compact valued, convex valued and upper hemicontinuous. Furthermore, $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{s}$ for all $\mathbf{x} \in \mathbf{d}_{\mathbf{c}}(\mathbf{p})$, where

$$\mathbf{s} = \sum_{h=1}^{n} \overline{\mathbf{x}}_h.$$

The economic interpretation of $\mathbf{d}_{\mathbf{c}}(\mathbf{p})$ is as the *aggregate demand* at prices $\mathbf{p}$ subject to rationing constraints $\mathbf{c}$. The economic interpretation of $\mathbf{s}$ is as the *aggregate supply*. This is a pure exchange economy, without production, so supply is independent of price.

Proof of Corollary 8.9

Proof of Corollary 8.9: That $\mathbf{d_c}$ is non-empty valued, compact valued, convex valued and upper hemicontinuous follows immediately from Propositions 8.7 and 8.8. $\hat{\xi}_{\mathbf{c},h}(\mathbf{p}) \subseteq B_{\mathbf{c}}(\mathbf{p}, \mathbf{p} \cdot \bar{\mathbf{x}}_h)$ so if $\mathbf{x}_h \in \hat{\xi}_{\mathbf{c},h}(\mathbf{p})$ then $\mathbf{p} \cdot \mathbf{x}_h \leq \mathbf{p} \cdot \bar{\mathbf{x}}_h$. Summing over h gives the inequality $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{s}$ for all $\mathbf{x} \in \mathbf{d_c}(\mathbf{p})$. This completes the proof of the corollary.

The following general theorem will be used to deduce the existence of equilibria in exchange economies.

Theorem 8.10: Suppose $K \subset \mathbf{R}^n$ is compact and $\zeta \colon \Delta \rightrightarrows K$ is non-empty valued, closed valued, convex valued and upper hemicontinuous. If $\mathbf{p} \cdot \mathbf{z} \leq 0$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in \zeta(\mathbf{p})$ then there is a $\mathbf{p}^* \in \Delta$ and a $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$ such that $\mathbf{z}^* \leq \mathbf{0}$.

Proof of Theorem 8.10

Proof of Theorem 8.10: Choose a compact convex set L containing K, e.g. a large closed ball. Let $\gamma(\mathbf{x}) = \max_{1\leq i\leq n} x_i$ and

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta\colon \mathbf{p}\cdot\mathbf{x} = \gamma(\mathbf{x})\}\,,$$

as in Proposition 8.3. We saw there that μ is non-empty valued, compact valued, convex valued and upper hemicontinuous. Also $\mathbf{p}\cdot\mathbf{x} \leq \mathbf{p}'\cdot\mathbf{x}$ for all $\mathbf{p}\in\Delta$ and $\mathbf{p}'\in\mu(\mathbf{x})$. Define $\Phi\colon \Delta\times L \rightrightarrows \Delta\times L$ by

$$\Phi(\mathbf{p},\mathbf{z}) = (\mu(\mathbf{z}), \zeta(\mathbf{p})).$$

μ and ζ are closed valued and upper hemicontinuous, so they have closed graphs by Proposition 2.11. Φ therefore is convex valued and has closed graph. By the Kakutani Fixed Point Theorem there is a $(\mathbf{p}^*,\mathbf{z}^*)\in\Delta\times L$ such that $\mathbf{p}^*\in\mu(\mathbf{z}^*)$ and $\mathbf{z}^*\in\zeta(\mathbf{p}^*)$. Now $\mathbf{p}^*\cdot\mathbf{z}\leq 0$ for all $\mathbf{z}\in\zeta(\mathbf{p}^*)$. Also $\mathbf{p}^*\in\mu(\mathbf{z}^*)$ implies $\mathbf{p}\cdot\mathbf{z}^*\leq\mathbf{p}^*\cdot\mathbf{z}$ for all $\mathbf{p}\in\Delta$. So $\mathbf{p}\cdot\mathbf{z}^*\leq 0$ for all $\mathbf{p}\in\Delta$. This is only possible if $\mathbf{z}^*\leq\mathbf{0}$. This completes the proof of the theorem.