MAU 34804 Lecture 26

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26 March 2020

Hemicontinuity of the budget correspondence

Last time we saw that the budget correspondence defined by
$B\colon \mathbf{R}^n_+ \times \mathbf{R}_+ \rightrightarrows \mathbf{R}^n_+$

$$B(\mathbf{p}, w) = \left\{\mathbf{x} \in \mathbf{R}^n_+ \colon \mathbf{p} \cdot \mathbf{x} \leq w\right\}.$$

is neither upper nor lower hemicontinuous. We did however have a partial substitute for hemicontinuity of B:

Proposition 8.1: Suppose $\mathbf{c} \in \mathbf{R}^n_+$ and $\mathbf{c} \gg \mathbf{0}$. Define
$B_{\mathbf{c}}\colon \mathbf{R}^n_+ \times \mathbf{R}_+ \rightrightarrows \mathbf{R}^n_+$ by

$$B_{\mathbf{c}}(\mathbf{p}, w) = \left\{\mathbf{x} \in \mathbf{R}^n_+ \colon \mathbf{x} \leq \mathbf{c}, \mathbf{p} \cdot \mathbf{x} \leq w\right\}.$$

Then $B_{\mathbf{c}}$ is non-empty valued, compact valued, convex valued and upper hemicontinuous, and its restriction to the set

$$\hat{\Gamma}^n = \left\{(\mathbf{p}, w) \in \mathbf{R}^n_+ \times \mathbf{R}_+ \colon w > 0\right\}$$

is lower hemicontinuous.

Another restricted budget correspondence

Proposition 8.2: B, defined above, is both upper and lower hemicontinuous when restricted to the set

$$\Gamma^n = \left\{(\mathbf{p}, w) \in \mathbf{R}^n_+ \times \mathbf{R}_+ \colon p \gg 0, w > 0\right\}.$$

$B|_{\Gamma^n}$ is also non-empty valued, compact valued and convex valued.
Proof: If $(\mathbf{p}', w') \in \Gamma^n$ then $p'_i > 0$ for all i. Choose $\mathbf{c}$ such that $c_i > w'/p'_i$ for all i. Let

$$N = \{(\mathbf{p}, w) \in \Gamma^n \colon p_i > w/c_i\}.$$

N is open and $(\mathbf{p}', w') \in N$. If $(\mathbf{p}, w) \in N$ and $\mathbf{x} \in B(\mathbf{p}, w)$ then $\mathbf{x} \geq 0$, $\mathbf{p} \cdot \mathbf{x} \leq w$ and $w > 0$. Then $p_i x_i \leq w < p_i c_i$ so $x_i < c_i$. So $B(\mathbf{p}, w) = B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$. By Proposition 8.1. $B_{\mathbf{c}}$ is non-empty valued, compact valued and convex valued and both upper and lower hemicontinuous on N, so the same is true of B. Since this holds in a neighbourhood N of each point $(\mathbf{p}', w') \in \Gamma^n$ the correspondence B is non-empty valued, compact valued and convex valued and both upper and lower hemicontinuous on Γ^n. This completes the proof of Proposition 8.2.

Proposition 8.3

The following fact will be used in the proof of Theorem 8.10.
Proposition 8.3: Define $\Delta \subseteq \mathbf{R}^n_+$, $\gamma\colon \mathbf{R}^n \to \mathbf{R}$ and $\mu\colon \mathbf{R}^n \rightrightarrows \Delta$ by

$$\Delta = \left\{ \mathbf{p} \in \mathbf{R}^n \colon \mathbf{p} \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}$$

$$\gamma(\mathbf{x}) = \max_{1 \leq i \leq n} x_i, \qquad \mu(\mathbf{x}) = \{ \mathbf{p} \in \Delta \colon \mathbf{p} \cdot \mathbf{x} = \gamma(\mathbf{x}) \}.$$

Then μ is non-empty valued, compact valued, convex valued upper hemicontinuous and $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p}' \cdot \mathbf{x} = \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$, $\mathbf{p}' \in \mu(\mathbf{x})$.
Proof: Suppose $\mathbf{p} \in \Delta$ and $\mathbf{x} \in \mathbf{R}^n$.

$$\mathbf{p} \cdot \mathbf{x} = \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} p_i \gamma(\mathbf{x}) = \gamma(\mathbf{x}),$$

with equality if and only if $p_i = 0$ whenever $x_i < \gamma(\mathbf{x})$.

Proof of Proposition 8.3, continued

Let

$$I(\mathbf{x}) = \left\{ i \in \{1, \ldots, n\} \colon x_i = \gamma(\mathbf{x}) = \max_{1 \le j \le n} x_j \right\}.$$

$$J(\mathbf{x}) = \left\{ i \in \{1, \ldots, n\} \colon x_i = \gamma(\mathbf{x}) < \max_{1 \le j \le n} x_j \right\}.$$

Then

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta \colon \forall j \in J(\mathbf{x}) \colon p_j = 0\}$$

$\mu(\mathbf{x})$ is clearly non-empty, compact and convex. Also $\mathbf{p} \cdot \mathbf{x} \le \gamma(\mathbf{x}) = \mathbf{p}' \cdot \mathbf{x}$. Suppose $\mathbf{x}' \in \mathbf{R}^n$. Then there is a $\theta \in \mathbf{R}$ such that $x'_j < \theta < \gamma(\mathbf{x}')$ for all $j \in J(\mathbf{x}')$. Let

$$N = \left\{ \mathbf{x} \in \mathbf{R}^n \colon \forall i \in I(\mathbf{x}') \colon x_i > \theta, \forall j \in J(\mathbf{x}') \colon x_j < \theta \right\}$$

N is an open neighbourhood of $\mathbf{x}'$ and $J(\mathbf{x}') \subseteq J(\mathbf{x})$ so $\mu(\mathbf{x}) \subseteq \mu(\mathbf{x}')$. If $\mu(\mathbf{x}') \subseteq V$ then $\mu(\mathbf{x}) \subseteq V$, so μ is upper hemicontinuous. This completes the proof of Proposition 8.3.

Proposition 8.4

A local utility maximiser within $B(\mathbf{p}, w)$ is a global utility maximiser.
Proposition 8.4: Suppose $\mathbf{p} \in \mathbf{R}^n_+$, $\mathbf{p} \neq \mathbf{0}$, $w > 0$, and $u\colon \mathbf{R}^n_+ \to \mathbf{R}$ is continuous, strictly increasing and quasiconcave. Let $B\colon \mathbf{R}^n_+ \times \mathbf{R}_+ \rightrightarrows \mathbf{R}^n_+$ be as before. If $\mathbf{x}^* \in B(\mathbf{p}, w)$ and there is an open $N \subseteq \mathbf{R}^n_+$ with $\mathbf{x}^* \in N$ such that $u(\mathbf{x}) \leq u(\mathbf{x}^*)$ for all $\mathbf{x} \in B(\mathbf{p}, w) \cap N$ then $\mathbf{p} \cdot \mathbf{x}^* = w$ and $u(\mathbf{x}) \leq u(\mathbf{x}^*)$ for all $\mathbf{x} \in B(\mathbf{p}, w)$.
Proof: We begin by noting that if $\mathbf{x} \in N \cap B(\mathbf{p}, w)$, $u(\mathbf{x}) \geq u(\mathbf{x}^*)$ then $\mathbf{p} \cdot \mathbf{x} = w$. To see this, choose $\mathbf{v} \in \mathbf{R}^n_+$ such that $\mathbf{p} \cdot \mathbf{v} > 0$ and let $\mathbf{r}(t) = \mathbf{x} + t\mathbf{v}$. $u \circ \mathbf{r}$ is strictly increasing because u is strictly increasing. Also, $\mathbf{r}(t) \in N$ for t near 0. So for small positive t, $\mathbf{r}(t) \in N$ and $u(\mathbf{r}(t)) > u(\mathbf{x}) \geq u(\mathbf{x}^*)$. Therefore $\mathbf{r}(t) \notin B(\mathbf{p}, w)$ for such t, i.e. $\mathbf{p} \cdot \mathbf{x} + t\mathbf{p} \cdot \mathbf{v} > w$. This implies $\mathbf{p} \cdot \mathbf{x} = w$. This holds in particular for $\mathbf{x} = \mathbf{x}^*$.

Proof of Proposition 8.4, continued

Suppose now that there is an $\mathbf{x}' \in B(\mathbf{p}, w)$ with $u(\mathbf{x}') > u(\mathbf{x}^*)$. $w > 0$ so there are points $\mathbf{x}'' \in B(\mathbf{p}, w)$ with $\mathbf{p} \cdot \mathbf{x}'' < w$. Choose one. Define

$$\mathbf{q}(s,t) = (1-t)\mathbf{x}^* + t\left[(1-s)\mathbf{x}' + s\mathbf{x}''\right]$$

for $s, t \in [0,1]$. $u(\mathbf{q}(0,1)) = u(\mathbf{x}') > u(\mathbf{x}^*)$, so, by continuity, $u(\mathbf{q}(s,1)) > u(\mathbf{x}^*)$ for small enough positive values of s. For such values of s we then have $\min(q(s,0), u(q(s,1))) = u(\mathbf{x}^*)$ and so $u(q(s,t)) \geq u(\mathbf{x}^*)$ for all $t \in (0,1)$, by the quasiconcavity of u. Because N is open and $\mathbf{q}(s,0) = \mathbf{x}^* \in N$ there is such a t that $\mathbf{q}(s,t) \in N$. Let $\mathbf{x} = \mathbf{q}(s,t)$, with s and t chosen as above. Then $\mathbf{x} \in N$, $u(\mathbf{x}) \geq u(\mathbf{x}^*)$ and $\mathbf{p} \cdot \mathbf{x} < w$. The last of these follows from $\mathbf{p} \cdot \mathbf{x}^* \leq w$, $\mathbf{p} \cdot \mathbf{x}' \leq w$ and $\mathbf{p} \cdot \mathbf{x}'' < w$. But we've already seen that this is impossible, we can't have $u(\mathbf{x}') > u(\mathbf{x}^*)$ for any $\mathbf{x}' \in B(\mathbf{p}, w)$. This completes the proof of the proposition.

Individual demand

The indirect utility function of a household is the maximum utility they can attain for given prices and wealth, as a function of those prices and wealth. Their demand correspondence gives the affordable bundles of goods which achieve that maximum. The following theorem describes that function and correspondence, assuming that prices and wealth are positive.

Proposition 8.5: Suppose $u\colon \mathbf{R}^n_+ \to \mathbf{R}$ is continuous, strictly increasing and quasiconcave. Define $V\colon \Gamma^n \to \mathbf{R}$ and $\xi\colon \Gamma^n \rightrightarrows \mathbf{R}^n_+$ by

$$V(\mathbf{p}, w) = \max_{\mathbf{x} \in B(\mathbf{p}, w)} u(\mathbf{x}),$$

$$\xi(\mathbf{p}, w) = \{\mathbf{x} \in B(\mathbf{p}, w) \colon u(\mathbf{x}) = V(\mathbf{p}, w)\}.$$

Then V is continuous and ξ is non-empty valued, compact valued, convex valued and upper hemicontinuous.

Proof of Proposition 8.5

Proof: By Proposition 8.2 the correspondence $B|_{\Gamma^n}$ satisfies all the properties required for the Berge Maximum Theorem (2.23), so V is continuous and ξ is non-empty valued, compact valued and upper hemicontinuous. Suppose that $\mathbf{x}, \mathbf{y} \in \xi(\mathbf{p}, w)$ and $t \in [0, 1]$. Then $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, w)$ and so $(1-t)\mathbf{x} + t\mathbf{y} \in B(\mathbf{p}, w)$. Also, using the quasiconcavity of u,

$$u((1-t)\mathbf{x} + t\mathbf{y}) \geq \min(u(\mathbf{x}), u(\mathbf{y})) = V(\mathbf{p}, w).$$

But $u(\mathbf{z}) \leq V(\mathbf{p}, w)$ for all $\mathbf{z} \in B(\mathbf{p}, w)$, so $u((1-t)\mathbf{x} + t\mathbf{y}) = V(\mathbf{p}, w)$ and $(1-t)\mathbf{x} + t\mathbf{y} \in \xi(\mathbf{p}, w)$. So ξ is convex valued. This completes the proof of the proposition.
We didn't actually use the assumption that u is strictly increasing.
We can drop the positive price assumption if we introduce rationing.

The demand correspondence with rationing

Proposition 8.6: Suppose $u\colon \mathbf{R}^n_+ \to \mathbf{R}$ is continuous, strictly increasing and quasiconcave $\mathbf{c} \gg \mathbf{0}$. Define $\hat{\Gamma}^n \subseteq \mathbf{R}^n_+ \times \mathbf{R}_+$, $V_{\mathbf{c}}\colon \hat{\Gamma}^n \to \mathbf{R}$ and $\xi_{\mathbf{c}}\colon \hat{\Gamma}^n \rightrightarrows \mathbf{R}^n_+$ by

$$\hat{\Gamma}^n = \left\{(\mathbf{p}, w) \in \mathbf{R}^n_+ \times \mathbf{R}_+ \colon w > 0\right\},$$

$$V_{\mathbf{c}}(\mathbf{p}, w) = \max_{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)} u(\mathbf{x}),$$

$$\xi_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w) \colon u(\mathbf{x}) = V(\mathbf{p}, w)\}.$$

Then $V_{\mathbf{c}}$ is continuous and $\xi_{\mathbf{c}}$ is non-empty valued, compact valued, convex valued and upper hemicontinuous.
The proof is the same as for Proposition 8.5, except with $\hat{\Gamma}^n$ in place of Γ^n, $B_{\mathbf{c}}$ in place of B, $V_{\mathbf{c}}$ in place of V, $\xi_{\mathbf{c}}$ in place of ξ, and Proposition 8.1 in place of Proposition 8.2.