MAU 34804 Lecture 25

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Exchange economies

The rest of the module is concerned with exchange economies. These are economic models where goods are exchanged, but not produced. This is complementary to models like Leontief's input-output model, which focus on production, ignoring issues of distribution. We'll see how individual preferences give rise to supply and demand "curves", and how these intersect at a market-clearing price.

We'll consider a market with m households and n goods. Neither the word "household" nor the word "good" is to be understood too narrowly. Quantities of goods are assumed to be non-negative real numbers, ignoring the fact that some are constrained to be integers. Each household has an initial endowment of goods and its own utility function.

A useful criterion for hemicontinuity

Lemma: $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous if and only if for every open $V \subseteq Y$ and $\mathbf{x} \in \Phi^+(V)$ there is an open $N \subseteq X$ such that $\mathbf{x} \in N$ and $N \subseteq \Phi^+(V)$ and Φ is lower hemicontinuous if and only if for every open $V \subseteq Y$ and every $\mathbf{x} \in \Phi^-(V)$ there is an open $N \subseteq X$ such that $\mathbf{x} \in N$ and $N \subseteq \Phi^-(V)$.
Proof: We've already seen that Φ is upper hemicontinuous if and only if $\Phi^+(V) = \{\mathbf{x} \in X\colon \Phi(\mathbf{x}) \subseteq V\}$ is open for all open $V \subset Y$ and is lower hemicontinuous if and only if $\Phi^-(V) = \{\mathbf{x} \in X\colon V \cap \Phi(\mathbf{x}) \neq \emptyset\}$ is open for all open $V \subset Y$. So the "only if" part of the lemma is trivial: we can just take $N = \Phi^\pm(V)$ for every $\mathbf{x} \in X$. The "if" part isn't much harder. If there is such an N for every $\mathbf{x} \in \Phi^\pm(V)$ then their union is open and is equal to $\Phi^\pm(V)$. This completes the proof of the lemma.

Notation

For $\mathbf{x}, \mathbf{y} \in \mathbf{R}^k$ we write $\mathbf{x} \leq \mathbf{y}$ when $x_i \leq y_i$ for all $i \in \{0, 1, \ldots, k\}$ and $\mathbf{x} \ll \mathbf{y}$ when $x_i < y_i$ for all such i. A function $u \colon X \to Y$, where $X \subseteq \mathbf{R}^k$ and $Y \subseteq \mathbf{R}$, is said to be strictly increasing if

$$\mathbf{x} \leq \mathbf{x}', \mathbf{x} \neq \mathbf{x}' \Rightarrow u(\mathbf{x}) < u(\mathbf{x}').$$

Also we define,

$$\mathbf{R}_+ = \{x \in \mathbf{R} \colon x \geq 0\}, \quad \mathbf{R}_+^k = (\mathbf{R}_+)^k = \left\{\mathbf{x} \in \mathbf{R}^k \colon \mathbf{x} \geq \mathbf{0}\right\}.$$

Note that these are not strict inequalities.
Examples of vectors which are naturally restricted to $\mathbf{R}_+^n$ would be $\mathbf{p}$, the vector whose j'th component is the price of the j'th good, or $\bar{\mathbf{x}}_h$, whose j'th component is the initial endowment of the j'th good held by the h'th household.

The budget correspondence

$\mathbf{x}$ represents a bundle of goods which can be bought by a household with wealth $w \in \mathbf{R}_+$ at prices $\mathbf{p}$ if and only if $\mathbf{p} \cdot \mathbf{x} \leq w$, i.e if and only if $\mathbf{x} \in B(\mathbf{p}, w)$, where

$$B(\mathbf{p}, w) = \left\{ \mathbf{x} \in \mathbf{R}^n_+ \colon \mathbf{p} \cdot \mathbf{x} \leq w \right\}.$$

Unfortunately B is neither upper nor lower hemicontinuous. If $n = 2$,

$$U = \left\{ \mathbf{x} \in \mathbf{R}^2_+ \colon x_1 < 1 + \frac{1}{1 + x_2^2} \right\},$$

$$V = \left\{ \mathbf{x} \in \mathbf{R}^2_+ \colon x_2 > 1 \right\}$$

then U and V are open, but $\Phi^+(U)$ and $\Phi^-(V)$ are not open. Details may be found in the notes.

A restricted budget correspondence

The budget correspondence with rationing is better behaved.

Proposition 8.1: Suppose $\mathbf{c} \in \mathbf{R}^n_+$ and $\mathbf{c} \gg \mathbf{0}$. Define $B_{\mathbf{c}} \colon \mathbf{R}^n_+ \times \mathbf{R}_+ \rightrightarrows \mathbf{R}^n_+$ by

$$B_{\mathbf{c}}(\mathbf{p}, w) = \left\{ \mathbf{x} \in \mathbf{R}^n_+ \colon \mathbf{x} \leq \mathbf{c}, \mathbf{p} \cdot \mathbf{x} \leq w \right\}.$$

Then $B_{\mathbf{c}}$ is non-empty valued, compact valued, convex valued and upper hemicontinuous, and its restriction to the set

$$\hat{\Gamma}^n = \left\{ (\mathbf{p}, w) \in \mathbf{R}^n_+ \times \mathbf{R}_+ \colon w > 0 \right\}$$

is lower hemicontinuous.

Proof: $B_{\mathbf{c}}(\mathbf{p}, w)$ is non-empty, because it contains $\mathbf{0}$, compact, because it is closed and bounded, and convex, because it is described by linear inequalities. Hemicontinuity requires more work. By the lemma we need to show that for all open $V \subseteq \mathbf{R}^n_+$ and $(\mathbf{p}', w') \in B^{\pm}_{\mathbf{c}}(V)$ there is an open $N \subseteq B^{\pm}_{\mathbf{c}}(V)$ with $(\mathbf{p}, w) \in N$.

Upper hemicontinuity

Let $C = \left\{ \mathbf{x} \in \mathbf{R}^n_+ \colon \mathbf{x} \leq \mathbf{c} \right\}$. Either $C \subseteq V$ or $C \not\subseteq V$. If $C \subseteq V$ then $B^+_{\mathbf{c}}(V) = \mathbf{R}^n_+$, which is open in itself, so we can just take $N = \mathbf{R}^n_+$. Suppose then that $V \not\subseteq C$ and $(\mathbf{p}', w') \in B^+_{\mathbf{c}}(V)$. Then $F = C - V$ is non-empty and compact, and $\mathbf{p}' \cdot \mathbf{x} > w'$ for all $\mathbf{x} \in F$. By the Extreme Value Theorem there is an $\mathbf{x}'' \in F$ such that $\mathbf{p}' \cdot \mathbf{x}'' \leq \mathbf{p}' \cdot \mathbf{x}$ for all $\mathbf{x} \in F$. Set $w'' = \mathbf{p}' \cdot \mathbf{x}''$. Now $\mathbf{x}'' \in F$ so $w'' > w'$. Choose $w''' \in (w', w'')$. Then $B_{\mathbf{c}}(\mathbf{p}', w''') \subseteq V$. Let $I, J \subseteq \{1, \ldots n\}$ and $N \subseteq \mathbf{R}^n_+ \times \mathbf{R}_+$ be defined by

$$I = \left\{ i \in \{1, \ldots, n\} \colon p'_i > 0 \right\}, \quad J = \left\{ i \in \{1, \ldots, n\} \colon p'_i = 0 \right\},$$

$$N = \left\{ (\mathbf{p}, w) \in \mathbf{R}^n_+ \times \mathbf{R}_+ \colon \forall i \in I \colon w''' p_i > w p'_i \right\}$$

Then N is open and $(\mathbf{p}', w') \in N$. If $(\mathbf{p}, 0) \in N$ and $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, 0)$ then $\mathbf{p} \cdot \mathbf{x} = 0$ and $p_i > 0$ for all $i \in I$, so $x_i = 0$ for such i. Equivalently, if $x_j > 0$ then $j \in J$. So $\mathbf{p}' \cdot \mathbf{x} = 0$.

Upper hemicontinuity continued

Because $\mathbf{p}' \cdot \mathbf{x} = 0$,

$$\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}', 0) \subseteq B_{\mathbf{c}}(\mathbf{p}', w') \subseteq V.$$

If $(\mathbf{p}, w) \in \mathbf{N}$, $w > 0$ and $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)$ then $\mathbf{p} \cdot \mathbf{x} \leq w$ and

$$\mathbf{p}' \cdot \mathbf{x} = \sum_{i=1}^{n} p_i' x_i \leq \frac{w'''}{w} \sum_{i=1}^{n} p_i x_i = \frac{w'''}{w} \mathbf{p} \cdot \mathbf{x} \leq w'''.$$

So $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}', w''') \subseteq V$. So if $(\mathbf{p}, w) \in N$ and $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)$ then, whether $w = 0$ or $w > 0$, we have $\mathbf{x} \in V$. In other words, $B_{\mathbf{c}}(\mathbf{p}, w) \subseteq V$ for all $(\mathbf{p}, w) \in N$. Still another way to say this is $N \subseteq B_{\mathbf{c}}^{+}(V)$. So for every open $V \subseteq \mathbf{R}_{+}^{n}$ and $(\mathbf{p}', w') \in B_{\mathbf{c}}^{+}(V)$ there is an open $N \subseteq B_{\mathbf{c}}^{+}(V)$ with $(\mathbf{p}', w') \in N$. By the lemma, $B_{\mathbf{c}}$ is upper hemicontinuous.

Lower hemicontinuity

Suppose $(\mathbf{p}', w') \in \hat{\Gamma}^n \cap B_{\mathbf{c}}^-(V)$ so $V \cap B_{\mathbf{c}}(\mathbf{p}', w') \neq \emptyset$, i.e. there is an $\mathbf{x}' \in V$ with $\mathbf{x}' \leq \mathbf{c}$ and $\mathbf{p}' \cdot \mathbf{x}' \leq w'$. Define $\mathbf{f} \colon (0, \infty) \to \mathbf{R}^n_+$ by $\mathbf{f}(\alpha) = \alpha \mathbf{x}'$. $\mathbf{f}$ is continuous and $\mathbf{f}(1) = \mathbf{x}'$, so there is an $\alpha < 1$ such that $\alpha \in \mathbf{f}^{-1}(V)$. Choose any such α and set $\mathbf{x} = \alpha \mathbf{x}'$. Define $g \colon \hat{\Gamma}^n \to \mathbf{R}$ by

$$g(\mathbf{p}, w) = \frac{1}{w} \mathbf{p} \cdot \mathbf{x}'.$$

Then $g(\mathbf{p}', w') \leq 1 < \alpha^{-1}$, so $(\mathbf{p}', w')$ belongs to the open set

$$N = g^{-1}\left(-\infty, \alpha^{-1}\right).$$

If $(\mathbf{p}, w) \in N$ then $\mathbf{p} \cdot \mathbf{x} \leq w$ and $\mathbf{x} \leq \mathbf{c}$ so $\mathbf{x} \in V \cap B_{\mathbf{c}}(\mathbf{p}, w)$, which is therefore non-empty. So $(\mathbf{p}, w) \subseteq B_{\mathbf{c}}^-(V)$. Every $(\mathbf{p}', w') \in \hat{\Gamma}^n \cap B_{\mathbf{c}}^-(V)$ has an open neighbourhood N in $\hat{\Gamma}^n \cap B_{\mathbf{c}}^-(V)$, and $B_{\mathbf{c}}|_{\hat{\Gamma}^n}$ is lower hemicontinuous by the lemma. This completes the proof of Proposition 8.1.