MAU 34804 Lecture 23

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Generalising the minimax theorem

How can we generalise von Neumann's theorem?

- Allow n players in place of two.
- Replace standard simplices with arbitrary non-empty compact convex subsets of a Euclidean space.
- Replace a single payoff function with continuous utility functions for each player, which they seek to maximise.
- Replace the linearity of the payoff for one player, given the strategy chosen by the other player, with quasiconcavity of the the utility of one player, given the strategies chosen by the other players.

Mathematical formulation

An n-person game, as above, is specified by the following information:

1. Non-empty compact convex $S_i \subseteq \mathbf{R}^{m_i}$ for $1 \leq i \leq n$.
2. Continuous functions $u_i \colon S \to \mathbf{R}$ for $1 \leq i \leq n$, where $S = S_1 \times \cdots \times S_n$, which are quasiconcave in their i'th argument when the other arguments are fixed.

We will show that for any such $S_1, \ldots, S_n$ and $u_1, \ldots, u_n$ there are $\mathbf{x}_1^* \in S_1, \ldots, \mathbf{x}_n^* \in S_n$ such that for each i

$$u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right) \leq u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i^*, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right)$$

Such a $(\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*) \in S$ is called a *Nash equilibrium*.

Recovering the von Neumann theorem

Is this a true generalisation of the minimax theorem proved earlier? Mathematically, yes. To get von Neumann's theorem it suffices to take $n = 2$, $S_1 = \Delta_P$, $S_2 = \Delta_Q$, $u_1 = f$ and $u_2 = -f$. It's straightforward to check that the hypotheses on S_i and u_i are satisfied. Relabeling $\mathbf{x}_1$, $\mathbf{x}_1^*$, $\mathbf{x}_2$ and $\mathbf{x}_2^*$ as $\mathbf{p}$, $\mathbf{p}^*$, $\mathbf{q}$ and $\mathbf{q}^*$ we can rewrite

$$u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right) \le u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i^*, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right)$$

as

$$f(\mathbf{p}, \mathbf{q}^*) \le f(\mathbf{p}^*, \mathbf{q}^*) \qquad -f(\mathbf{p}^*, \mathbf{q}) \le -f(\mathbf{p}^*, \mathbf{q}^*)$$

or, equivalently,

$$f(\mathbf{p}, \mathbf{q}^*) \le f(\mathbf{p}^*, \mathbf{q}^*) \le f(\mathbf{p}^*, \mathbf{q}).$$

Difference of interpretation

Economically, it's not such a good generalisation. Recall that

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*)$$

means Player A will be at least as well off choosing $\mathbf{p}^*$ as any other available strategy, assuming Player B chooses $\mathbf{q}^*$, while

$$f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}).$$

means they'll be no worse off if Player B chooses some other available strategy, assuming they've chosen $\mathbf{p}^*$. The analogue of the two inequalities above in an n-person game would be

$$u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right) \leq u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i^*, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right),$$

$$u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i^*, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right) \leq u_i\left(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_i^*, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n\right)$$

We're going to get the first of these, but not the second.

Notation

As before, $S = S_1 \times \cdots \times S_n$. Set

$$S_{-i} = S_1 \times S_{i-1} \times S_{i+1} \times S_n.$$

There are natural projections $\pi_i \colon S \to S_i$ and $\pi_{-i} \colon S \to S_{-i}$ and a natural map $\sigma \colon S_i \times S_{-i} \to S$ such that $\sigma(\pi_i(\mathbf{x}), \pi_{-i}(\mathbf{x})) = \mathbf{x}$.
Define $b_i \colon S_{-i} \to \mathbf{R}$ and $B_i \colon S_{-i} \rightrightarrows S_i$ by

$$b_i\left(\mathbf{x}_{-i}\right) = \max_{\pi_{-i}(\mathbf{x}) = \mathbf{x}_{-i}} u_i(\mathbf{x}),$$

$$B_i(\mathbf{x}_{-i}) = \{\mathbf{x}_i \in S_i \colon u_i(\sigma(\mathbf{x}_i, \mathbf{x}_{-i})) = b_i(\mathbf{x}_{-i})\}$$

Define $\Phi \colon S \rightrightarrows S$ and $G_i \subseteq S \times S$ by

$$\pi_i \circ \Phi = B_i \circ \pi_{-i},$$

$$G_i = \{(\mathbf{x}, \mathbf{y}) \in S \times S \colon \pi_i(\mathbf{y}) \in B_i\left(\pi_{-i}(\mathbf{x})\right)\}$$

Proof of existence of Nash equilibria

b_i is continuous and B_i is non-empty valued, compact valued, convex valued and upper hemicontinuous by the Berge Maximum Theorem (2.23). $\mathrm{Graph}(B_i)$ is therefore closed by Proposition 2.11. $G_i = (\pi_i \times \pi_{-i})^{-1}(\mathrm{Graph}(B_i))$ is then closed because $\pi_i \times \pi_{-i} \colon S \times S \to S_i \times S_{-i}$ is continuous. $\mathrm{Graph}(\Phi) = \bigcap_{i=1}^{n} G_i$ is therefore closed. Φ is non-empty valued because B_i is non-empty valued for each i and the Cartesian product of non-empty sets is non-empty. If $\mathbf{x}_i \in S_i$ and $\mathbf{x}_{-i} \in S_{-i}$ then

$$u_i(\sigma(\mathbf{x}_i, \mathbf{x}_{-i})) \leq \max_{\pi_{-i}(\mathbf{x})=\mathbf{x}_{-i}} u_i(\mathbf{x}) = b_i(\mathbf{x}_{-i})$$

so

$$B_i(\mathbf{x}_{-i}) = \{\mathbf{x}_i \in S_i \colon u_i(\sigma(\mathbf{x}_i, \mathbf{x}_{-i})) \geq b_i(\mathbf{x}_{-i})\}$$

and hence, by Lemma 7.2, $B_i(\mathbf{x}_{-i})$ is convex. The product of convex sets is convex, so $\Phi(\mathbf{x})$ is convex for all $\mathbf{x} \in S$.

Conclusion of proof

We've just shown that $\Phi\colon S \rightrightarrows S$ is non-empty valued, convex valued and has closed graph. We can therefore apply Kakutani's Fixed Point Theorem (5.4) to get an $\mathbf{x}^*$ such that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$. Equivalently, $\mathbf{x}_i^* \in B_i(\mathbf{x}_{-i}^*)$, where $\mathbf{x}_i^* = \pi_i(\mathbf{x}^*)$ and $\mathbf{x}_{-i}^* = \pi_{-i}(\mathbf{x}^*)$. This means $u_i(\sigma(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)) = b_i(\mathbf{x}_{i-1}^*)$. Equivalently,

$$u_i(\sigma(\mathbf{x}_i, \mathbf{x}_{-i}^*)) \leq u_i(\sigma(\mathbf{x}_i^*, \mathbf{x}_{-i}^*))$$

for all $\mathbf{x}_i \in S_i$, or

$$u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right) \leq u_i\left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i^*, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right).$$

This completes the proof.

Statement of theorem

We've just proved

Theorem 7.3: for every collection of non-empty, compact convex $S_i \subseteq \mathbf{R}^{m_i}$ and continuous quasiconcave $u_i \colon S \to \mathbf{R}$ for $1 \leq i \leq n$, where $S = S_1 \times \cdots \times S_n$, there is a Nash equilibrium, i.e. a point $\mathbf{x}^* \in S$ such that

$$u_i \left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right) \leq u_i \left(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i^*, \mathbf{x}_{i+1}^*, \ldots, \mathbf{x}_n^*\right).$$

for all $1 \leq i \leq n$ and $\mathbf{x}_i \in S_i$.