MAU 34804 Lecture 22

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Zero sum two person games

We consider games between two players, each of whom has a finite number of pure stategies available. The net payment from one player to the other is a function of the pure strategies each selects. There are no other players or payments, so the sum of the net payment from Player A to Player B and the net payment from Player B to Player A is zero. In addition to the pure strategies players can adopt mixed strategies, selecting a pure strategy at random with specified probabalities. A mixed strategy is characterised by those probabilities. We assume the goal of each player is to maximise the expected value of the net payment they receive.

Mathematical formulation

Let m and n be the numbers of pure strategies available to Players A and B respectively. Fix an ordering of those strategies. A mixed strategy for Player A is then an element of the $m-1$-simplex

$$\Delta_P = \left\{ \mathbf{p} \in \mathbf{R}^m \colon p_1 \geq 0, \ldots, p_m \geq 0, \sum_{i=1}^{m} p_i = 1 \right\},$$

where p_i is to be interpreted as the probability that the player selects the i'th pure strategy. Similarly, Player B's mixed strategies are elements of

$$\Delta_Q = \left\{ \mathbf{q} \in \mathbf{R}^m \colon q_1 \geq 0, \ldots, q_n \geq 0, \sum_{j=1}^{n} q_j = 1 \right\}.$$

Note that pure strategies are also mixed strategies, just with all but one of their probabilities equal to zero. They correspond to the vertices of the simplices.

Optimal response

If a_{ij} is the net payment from Player B to Player A when Player A selects their i'th pure strategy and Player B selects their j'th then the expected net payment when they choose mixed strategies $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$ is

$$f(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \mathbf{p}^T A \mathbf{q}.$$

Suppose Player A has to choose their mixed strategy first, and this choice is revealed to Player B. If Player A chooses $\mathbf{p} \in \Delta_P$ then the best expected outcome Player B can achieve is

$$\mu_Q(\mathbf{p}) = \min_{\mathbf{q} \in \Delta_Q} f(\mathbf{p}, \mathbf{q}).$$

The minimum exists by the Extreme Value Theorem. To achieve it they should choose a mixed strategy $\mathbf{q}$ from

$$Q(\mathbf{p}) = \{\mathbf{q} \in \Delta_Q \colon f(\mathbf{p}, \mathbf{q}) = \mu_Q(\mathbf{p})\}.$$

Payoff from optimal strategies

Working backwards, if the Player A knows that Player B will respond optimally then they should choose a mixed strategy $\mathbf{p}$ which maximises $\mu_Q(\mathbf{p})$, to achieve an expected net payment of

$$\max_{\mathbf{p}\in\Delta_P} \mu_Q(\mathbf{p}) = \max_{\mathbf{p}\in\Delta_P} \min_{\mathbf{q}\in\Delta_Q} f(\mathbf{p},\mathbf{q}).$$

Similarly, if Player B chooses first then Player A should respond to $\mathbf{q}$ with $\mathbf{p} \in Q(\mathbf{p})$ where

$$P(\mathbf{q}) = \{\mathbf{p} \in \Delta_P \colon f(\mathbf{p},\mathbf{q}) = \mu_P(\mathbf{q})\}, \quad \mu_P(\mathbf{q}) = \max_{\mathbf{q}\in\Delta_Q} f(\mathbf{p},\mathbf{q}).$$

Knowing this, Player B chooses $\mathbf{q}$, achieving the expected payment

$$\min_{\mathbf{q}\in\Delta_P} \mu_P(\mathbf{q}) = \min_{\mathbf{q}\in\Delta_Q} \max_{\mathbf{p}\in\Delta_P} f(\mathbf{p},\mathbf{q}).$$

Von Neumann's Minimax Theorem

General properties of maxima and minima ensure that

$$\max_{\mathbf{p}\in\Delta_P} \min_{\mathbf{q}\in\Delta_Q} f(\mathbf{p},\mathbf{q}) \leq \min_{\mathbf{q}\in\Delta_Q} \max_{\mathbf{p}\in\Delta_P} f(\mathbf{p},\mathbf{q}).$$

In other words, there is nothing to be gained by being forced to choose your strategy first. Remarkably though, if both players play optimally there's no penalty to being forced to go first either.
Von Neumann's Minimax Theorem (7.1): With Δ_P, Δ_Q and f defined as above,

$$\max_{\mathbf{p}\in\Delta_P} \min_{\mathbf{q}\in\Delta_Q} f(\mathbf{p},\mathbf{q}) = \min_{\mathbf{q}\in\Delta_Q} \max_{\mathbf{p}\in\Delta_P} f(\mathbf{p},\mathbf{q}).$$

In fact there is a $(\mathbf{p}^*,\mathbf{q}^*) \in \Delta_P \times \Delta_Q$ such that for all $(\mathbf{p},\mathbf{q}) \in \Delta_P \times \Delta_Q$

$$f(\mathbf{p},\mathbf{q}^*) \leq f(\mathbf{p}^*,\mathbf{q}^*) \leq f(\mathbf{p}^*,\mathbf{q}).$$

Interpretation

What does

$$f(\mathbf{p},\mathbf{q}^*) \le f(\mathbf{p}^*,\mathbf{q}^*) \le f(\mathbf{p}^*,\mathbf{q}).$$

mean? From Player A's point of view,

$$f(\mathbf{p},\mathbf{q}^*) \le f(\mathbf{p}^*,\mathbf{q}^*)$$

means that they'll be at least as well off choosing $\mathbf{p}^*$ as any other available strategy, assuming that Player B chooses $\mathbf{q}^*$. But

$$f(\mathbf{p}^*,\mathbf{q}^*) \le f(\mathbf{p}^*,\mathbf{q}).$$

means that they'll be no worse off if Player B chooses some other available strategy, assuming they've chosen $\mathbf{p}^*$. From Player B's point of view things are similar, switching the roles of the players and their strategies and switching the roles of the two inequalities.

Proof of von Neumann's theorem

By Berge's Maximum Theorem (2.23) the correspondences P and Q are non-empty valued, compact valued, convex valued and upper hemicontinous. The same is then true of the correspondence $\Phi\colon \Delta_P \times \Delta_Q \rightrightarrows \Delta_P \times \Delta_Q$ defined by

$$\Phi(\mathbf{p}, \mathbf{q}) = (P(\mathbf{q}), Q(\mathbf{p})).$$

Every compact subset of $\mathbf{R}^{m+n}$ is closed, so Φ is closed valued and hence, by Proposition 2.11, $\mathrm{Graph}(\Phi)$ is closed. Also, products of convex sets are convex. By the Kakutani Fixed Point Theorem (5.4), there is a $(\mathbf{p}^*, \mathbf{q}^*) \in \Delta_P \times \Delta_Q$ such that $(\mathbf{p}^*, \mathbf{q}^*) \in \Phi(\mathbf{p}^*, \mathbf{q}^*)$, i.e. $\mathbf{p}^* \in P(\mathbf{q}^*)$ and $\mathbf{q}^* \in Q(\mathbf{p}^*)$. The first of these statements implies $f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*)$ for all $p \in \Delta_P$, while the second implies $f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$. for all $q \in \Delta_Q$. This completes the proof.

Quasiconvexity and quasiconcavity

The function f from von Neumann's theorem was linear in each of its arguments for fixed values of the other argument. For generalisations it's useful to weaken that property. If $K \in \mathbf{R}^n$ is convex then $f\colon K \to \mathbf{R}$ is called quasiconvex if

$$f\left((1-t)\mathbf{u} + t\mathbf{v}\right) \leq \max\left(f(\mathbf{u}), f(\mathbf{v})\right)$$

for all $t \in [0,1]$ and is called quasiconcave if

$$f\left((1-t)\mathbf{u} + t\mathbf{v}\right) \geq \min\left(f(\mathbf{u}), f(\mathbf{v})\right)$$

$$\min\left(f(\mathbf{u}), f(\mathbf{v})\right) \leq (1-t)f(\mathbf{u}) + tf(\mathbf{v}) \leq \max\left(f(\mathbf{u}), f(\mathbf{v})\right)$$

so convex functions are quasiconvex and concave functions are quasiconcave. Linear functions are quasiconvex and quasiconcave.
Lemma 7.2: $f^{-1}((-\infty, b])$ is convex if f is quasiconvex and $f^{-1}([a, \infty))$ is convex if f is quasiconcave.