MAU 34804 Lecture 21

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13 March 2020

The $\mathbf{c}$-norm

Recall from last lecture that A is a matrix with non-negative entries such that some power of A has positive entries. On that assumption we found vectors $\mathbf{b}$ and $\mathbf{c}$ in the interior of the simplex $\Delta_{\mathbf{w}}$ and a $\mu > 0$ such that $A\mathbf{b} = \mu\mathbf{b}$ and $\mathbf{c}^T A = \mu\mathbf{c}^T$. Define

$$\|\mathbf{x}\|_{\mathbf{c}} = \sum_{j=1}^{n} c_j |x_j|.$$

It is easily verified that this is a norm on either $\mathbf{R}^n$ or $\mathbf{C}^n$. Also,

$$\begin{aligned}\|A\mathbf{x}\|_{\mathbf{c}} &= \sum_{j=1}^{n} c_j \left|\sum_{k=1}^{n} a_{jk} x_k\right| \leq \sum_{j=1}^{n} c_j \sum_{k=1}^{n} a_{jk} \left|x_k\right| \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} c_j a_{jk} \left|x_k\right| = \mu \sum_{k=1}^{n} c_k \left|x_k\right| = \mu \|\mathbf{x}\|_{\mathbf{c}}.\end{aligned}$$

Then, by induction on l, $\|A^l \mathbf{x}\| \leq \mu^l \|\mathbf{x}\|_{\mathbf{c}}$.

Geometric series

If $|\lambda| > \mu$ then

$$\sum_{l=0}^{\infty} \|\lambda^{-l-1}A^l\mathbf{x}\|_{\mathbf{c}} \leq \sum_{l=0}^{\infty} |\lambda|^{-l-1}\mu^l\|\mathbf{x}\|_{\mathbf{c}} = \frac{\|\mathbf{x}\|_{\mathbf{c}}}{|\lambda| - \mu} < \infty.$$

So the series $\mathbf{y} = \sum_{l=0}^{\infty} \lambda^{-l-1}A^l\mathbf{x}$ converges absolutely in the $\mathbf{c}$-norm. Then

$$\begin{aligned}(\lambda I - A)\mathbf{y} &= \sum_{l=0}^{\infty} \lambda^{-l}A^l\mathbf{x} - \sum_{l=0}^{\infty} \lambda^{-l-1}A^{l+1}\mathbf{x} \\ &= \sum_{l=0}^{\infty} \lambda^{-l}A^l\mathbf{x} - \sum_{l=1}^{\infty} \lambda^{-l}A^l\mathbf{x} = \mathbf{x},\end{aligned}$$

so $\mathbf{y} = (\lambda I - A)^{-1}\mathbf{x}$. In particular the null space of $\lambda I - A$ is trivial and so λ is not an eigenvalue of A.

Monotonicity

Suppose now that λ is real and $\lambda > 0$. As we just saw,

$$(\lambda I - A)^{-1} = \sum_{l=0}^{\infty} \lambda^{-l-1} A^l.$$

Every term on the right hand side is a matrix with non-negative entries and at least one of them has positive entries, so all entries of $(\lambda I - A)$ are positive. Similarly,

$$\frac{d}{d\lambda}(\lambda I - A)^{-1} = -\sum_{l=0}^{\infty} (l+1)\lambda^{-l-2} A^l$$

and

$$\frac{d^2}{d\lambda^2}(\lambda I - A)^{-1} = \sum_{l=0}^{\infty} (l+1)(l+2)\lambda^{-l-3} A^l,$$

so all entries of $(\lambda I - A)^{-1}$ are strictly decreasing and strictly convex.

More monotonicity

We now reformulate the preceding results on the vector equation $\mathbf{y} = (\lambda I - A)^{-1}\mathbf{x}$ in terms of solutions of the equivalent system of scalar equations

$$\lambda y_i = x_i + \sum_{j=1}^{n} a_{ij} y_j.$$

This system has a unique solution for every $\lambda > \mu$ and $x_1 > 0$, $\ldots x_n > 0$. For this solution each y_i is positive and is a strictly increasing function of x_j for each j and is a strictly decreasing convex function of λ. Also, it's a strictly increasing function of a_{jk} for each j and k.

More about eigenvalues

We've already seen than there are no eigenvalues larger than μ. μ is itself an eigenvalue, so it is *an* eigenvalue of largest norm, but is it *the* eigevalue of largest norm? Yes! But it takes some work to prove this.

Let $V = \{\mathbf{x} \in \mathbf{R}^n \colon \mathbf{c}^T\mathbf{x} = 0\}$ and let W be the space spanned by the single vector $\mathbf{b}$. If $\mathbf{x} \in V \cap W$ then $\mathbf{x} = \alpha\mathbf{b}$ for some $\alpha \in \mathbf{R}$ and

$$0 = \mathbf{c}^T\mathbf{x} = \alpha \sum_{i=1}^{n} b_i c_i.$$

This implies $\alpha = 0$ because $\sum_{i=1}^{n} b_i c_i > 0$. So $V \cap W = \{\mathbf{0}\}$. But V is of dimension $n-1$ and W is of dimension 1 so $\mathbf{R}^n = V \oplus W$. Also $A(V) \subseteq V$, since $\mathbf{c}^T A\mathbf{x} = \mu\mathbf{c}^T\mathbf{x} = 0$ if $\mathbf{c}^T\mathbf{x} = 0$, and $A(W) \subseteq W$, since $A\alpha\mathbf{b} = \alpha\mu\mathbf{b}$.

Action on the simplex $\Delta_{\mathbf{c}}$

Recall the action of A on the simplex

$$\Delta_{\mathbf{w}} = \left\{ \mathbf{p} \in \mathbf{R}^n \colon p_0 \geq 0, \ldots, p_n \geq 0, \sum_{j=0}^{n} w_j p_j = 1 \right\}.$$

by $f_{\mathbf{w}} \colon \Delta_{\mathbf{w}} \to \Delta_{\mathbf{w}}$ by $f_{\mathbf{w}}(\mathbf{p}) = \mathbf{q}$, where

$$q_i = \frac{\sum_{k=1}^{n} a_{ik} p_k}{\sum_{j=1}^{n} \sum_{k=1}^{n} w_j a_{jk} p_k}$$

which we defined in the last lecture. We saw there that $f_{\mathbf{w}}$ maps $\Delta_{\mathbf{w}}$ into the *interior* of $\Delta_{\mathbf{w}}$. Now

$$f_{\mathbf{c}}(p) = \frac{1}{\mu} A\mathbf{p}$$

for $\mathbf{p} \in \Delta_{\mathbf{c}}$, so $\mu^{-1} A$ maps $\Delta_{\mathbf{c}}$ to its interior.

Brin's argument

Now $\Delta_{\mathbf{c}}$ belongs neither to V nor to W, but its translate

$$\Sigma = \left\{ \mathbf{x} \in \mathbf{R}^n \colon \mathbf{x} + \left(\mathbf{c}^T \mathbf{b} \right)^{-1} \mathbf{b} \in \Delta_{\mathbf{c}} \right\}$$

is a simplex of V. $\mu^{-1} A(\Sigma) \subseteq \Sigma$ and $\mathbf{0}$ belongs to the interior of V. We can therefore apply
Brin's Lemma: Suppose V is a real vector space, Σ is a simplex in V whose interior contains $\mathbf{0}$, $T \colon V \to V$ is linear and $T(\Sigma)$ lies in the interior of Σ. Then T has no eigenvalues, real or complex, with eigenvalue of absolute value 1. Assuming this lemma, which we will prove shortly, $\mu^{-1} A|_V$ has eigenvalues less than 1, so the largest eigenvalue of A is μ and this eigenvalue has algebraic and geometric multiplicity 1.

Proof of Brin's Lemma

Suppose $\mathbf{x} \in V$ is an eigenvector of T with eigenvalue $\nu = e^{2\pi i\alpha}$, either real or complex. Let S be the subspace spanned by the real and complex parts of $\mathbf{x}$. S is of dimension either 1 or 2. If α is rational then there is an infinite sequence of powers of ν such which are equal to 1. If α is irrational then there is an infinite sequence of powers of ν which tends to 1. So in either case there is an increasing sequence $j_1, j_2, \ldots$ of positive integers such that $\nu^{j_k} \to 1$ as $k \to \infty$. If $\mathbf{y} \in S \cap \partial\Sigma$ then

$$T^{j_k} = \frac{\nu^{j_k} + \overline{\nu}^{j_k}}{2} I + \frac{\nu^{j_k} - \overline{\nu}^{j_k}}{\nu - \overline{\nu}} T \to \mathbf{y}$$

But $T^{j_k}\mathbf{y} \in T(\Sigma)$ and $T(\Sigma)$ is closed, so $\mathbf{y} \in T(\Sigma)$. $T(\Sigma)$ though lies in the interior of Σ, not the boundary, so we have a contradiction. This completes the proof of the lemma.

Leontief

Wassily Leontief won the "Nobel Prize in Economics" for a model with n industries producing output some of which is to be used as input for other industries and some for consumption. a_{ij} is the amount of industry i's output that industry j needs as an input to produce a unit of its own output, x_i is the amount of industry i's output produced and y_i is the amount of its output consumed by consumers. Assume that $x_i \geq 0$, $y_i \geq 0$ and $a_{ij} > 0$. The equation

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + y_i$$

expresses the fact that all of industry i's output goes either towards producing inputs of other industries or towards consumption. This system has a unique solution $\mathbf{x}$ for given $\mathbf{y}$ if and only if all eigenvalues of A are of absolute value less than 1.