MAU 34804 Lecture 20

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Kakutani Fixed Point Theorem

Theorem 5.4: Suppose $X \subseteq \mathbf{R}^n$ is a non-empty compact convex subset of a Euclidean space and $\Phi\colon X \rightrightarrows X$ is non-empty valued, convex valued and has closed graph. Then Φ has a fixed point, i.e. there is an $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Proof: X is bounded, so there is an n-simplex Δ such that $X \subseteq \Delta$. By Proposition 3.8 there is a retraction $r\colon \mathbf{R}^n \to X$. Its restriction, which we'll also call r, is a retraction $r\colon \Delta \to X$. Define $\Psi\colon \Delta \rightrightarrows \Delta$ by

$$\Psi(\mathbf{x}) = \Phi(r(\mathbf{x}))$$

for $\mathbf{x} \in \Delta$. Like Φ, Ψ is non-empty valued, convex valued and has closed graph. Let K be the simplical complex consisting of Δ and its faces. Let $K^{(j)}$ be a sequence of simplicial complexes such that $K^{(0)} = K$, $K^{(j+1)}$ is a subdivision of $K^{(j)}$ and $\mu(K^{(j)}) \to 0$. These could, for example, be the successive barycentric subdivisions.

Proof of Kakutani, continued

For each $\mathbf{v} \in \bigcup_{j=0}^{\infty} \mathrm{Vert}(K^{(j)})$ choose a $\psi(\mathbf{v}) \in \Psi(\mathbf{v})$. We know there is at least one such choice because Φ, and therefore Ψ, is non-empty valued. Define a piecewise linear $f_j \colon \Delta \to \Delta$ by choosing $f_j(\mathbf{v}) = \psi(\mathbf{v})$ if $\mathbf{v} \in \mathrm{Vert}(K^{(j)})$ and f_j is linear on each simplex of $K^{(j)}$. There is a unique such f_j by Proposition 4.8 and it is continuous by Proposition 4.7. By the Brouwer Fixed Point Theorem (5.3) there is a $\mathbf{z}_j \in \Delta$ such that $f_j(\mathbf{z}_j) = \mathbf{z}_j$. This $\mathbf{z}_j$ belongs to some n-simplex $\sigma_j \in K^{(j)}$ and has barycentric coordinates $t_{i,j}$ with respect to the vertices $\mathbf{v}_{i,j}$ of σ_j. Set

$$\mathbf{y}_{i,j} = f_j(\mathbf{v}_{i,j}) = \psi(\mathbf{v}_{i,j}) \in \Psi(\mathbf{v}_{i,j}).$$

Then $\sum_{i=0}^{n} t_{i,j} = 1$ and $\mathbf{z}_j = \sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j}$ is equal to

$$f_j(\mathbf{z}_j) = f_j\left(\sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j}\right) = \sum_{i=0}^{n} t_{i,j} f\left(\mathbf{v}_{i,j}\right) = \sum_{i=0}^{n} t_{i,j}\mathbf{y}_{i,j}$$

Still more Kakutani

Now $(t_{0,j},\ldots,t_{n,j},\mathbf{v}_{0,j},\ldots,\mathbf{v}_{n,j},\mathbf{y}_{0,j},\ldots,\mathbf{y}_{n,j}) \in [0,1]^{n+1}\times\Delta^{2n+2}$, which is a bounded subset of $\mathbf{R}^{2n^2+3n+1}$. By Bolzano-Weierstrass (1.4) there is a subsequence converging to some point $(t_{0,\infty},\ldots,t_{n,\infty},\mathbf{v}_{0,\infty},\mathbf{v}_{n,\infty},\mathbf{y}_{0,\infty},\mathbf{y}_{n,\infty})$. Taking limits in the equations

$$\sum_{i=0}^{n} t_{i,j} = 1 \qquad \sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j} = \sum_{i=0}^{n} t_{i,j}\mathbf{y}_{i,j}$$

gives

$$\sum_{i=0}^{n} t_{i,\infty} = 1 \qquad \sum_{i=0}^{n} t_{i,\infty}\mathbf{v}_{i,\infty} = \sum_{i=0}^{n} t_{i,\infty}\mathbf{y}_{i,\infty}$$

Because $\mu(K^{(j)}) \to 0$ the $\mathbf{v}_{i,\infty}$ are all equal. Call their common value $\mathbf{x}^*$.

Proof of Kakutani, concluded

We now have

$$\sum_{i=0}^{n} t_{i,\infty} = 1 \qquad \mathbf{x}^* = \sum_{i=0}^{n} t_{i,\infty} \mathbf{y}_{i,\infty}.$$

Also,

$$(\mathbf{v}_{i,j}, \mathbf{y}_{i,j}) \in \mathrm{Graph}(\Psi)$$

and $\mathrm{Graph}(\Psi)$ is closed so, taking limits,

$$(\mathbf{x}^*, \mathbf{y}_{i,\infty}) \in \mathrm{Graph}(\Psi),$$

i.e. $\mathbf{y}_{i,\infty} \in \Psi(\mathbf{x}^*)$. But Ψ is convex valued, so

$$\mathbf{x}^* \in \Psi(\mathbf{x}^*).$$

Finally, note that $\Psi(\mathbf{x}^*) = \Phi(r(\mathbf{x}^*)) \subseteq X$, so $\mathbf{x}^* \in X$. This concludes the proof.

Perron Matrices

Suppose A is an $n \times n$ matrix, that every entry in A is non-negative and that there is some positive integer m such that every entry of A^m is positive. This condition holds in particular if all entries of A are positive, but holds also for some matrices not all of whose entries are positive, such as

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

No row or column of A can be identically zero, since otherwise the corresponding row or column of A^m would be identically zero. We'll be interested in the action of A on the simplex

$$\Delta_{\mathbf{w}} = \left\{ \mathbf{p} \in \mathbf{R}^n \colon p_0 \geq 0, \ldots, p_n \geq 0, \sum_{j=0}^{n} w_j p_j = 1 \right\}.$$

where $w_j > 0$ for each j.

Action on the simplex $\Delta_{\mathbf{w}}$

$\frac{a_{ik}}{w_k} \geq \min_{1 \leq j \leq n} \frac{a_{ij}}{w_j}$ so

$$\sum_{k=1}^{n} a_{ik} p_k \geq \sum_{k=1}^{n} \min_{1 \leq j \leq n} \frac{a_{ij}}{w_j} w_k p_k = \min_{1 \leq j \leq n} \frac{a_{ij}}{w_j} > 0.$$

if $\mathbf{p} \in \Delta_{\mathbf{w}}$. Here we've used the fact that $\min_{1 \leq j \leq n} \frac{a_{ij}}{w_j}$ is independent of k and $\sum_{k=1}^{n} w_k p_k = 1$. We can therefore define a function $f_{\mathbf{w}} \colon \Delta_{\mathbf{w}} \to \Delta_{\mathbf{w}}$ by $f(\mathbf{p}) = \mathbf{q}$, where

$$q_i = \frac{\sum_{k=1}^{n} a_{ik} p_k}{\sum_{j=1}^{n} \sum_{k=1}^{n} w_j a_{jk} p_k}.$$

$q_i > 0$ so $\mathbf{q}$ belongs to the interior of $\Delta_{\mathbf{w}}$.

The fixed point

The function $f_{\mathbf{w}}$ just defined is continuous, so by the Brouwer Fixed Point Theorem (5.3) there is a $\mathbf{b} \in \Delta_{\mathbf{w}}$ such that $f_{\mathbf{w}}(\mathbf{b}) = \mathbf{b}$. Equivalently,

$$\mu b_i = \sum_{k=1}^{n} a_{ik} b_k$$

where

$$\mu = \sum_{j=1}^{n} \sum_{k=1}^{n} w_j a_{jk} b_k > 0.$$

In terms of matrices, $A\mathbf{b} = \mu\mathbf{b}$, i.e. $\mathbf{b}$ is an eigenvector of A with eigenvalue μ. We've already seen that $f(\mathbf{b})$ lies in the interior of $\Delta_{\mathbf{w}}$, so $b_i > 0$.

Duality

We could apply the same analysis to A^T in place of A to get a $\mathbf{c}$ with all entries positive and a $\nu > 0$ such that

$$\nu c_k = \sum_{k=1}^{n} c_i a_{ik}, \qquad \nu = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j a_{jk}.$$

What is the relation between μ and ν?

$$\mu \sum_{i=1}^{n} c_i b_i = \sum_{i=1}^{n} c_i \sum_{k=1}^{n} a_{ik} b_k = \sum_{i=1}^{n} \sum_{k=1}^{n} c_i a_{ik} b_k = \sum_{k=1}^{n} \sum_{i=1}^{n} c_i a_{ik} b_k$$

$$\nu \sum_{i=1}^{n} c_i b_i = \nu \sum_{k=1}^{n} c_k b_k = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} c_i a_{ik} \right) b_k = \sum_{k=1}^{n} \sum_{i=1}^{n} c_i a_{ik} b_k$$

$\sum_{i=1}^{n} c_i b_i > 0$ so we can conclude that $\mu = \nu$. What is the relation between $\mathbf{b}$ and $\mathbf{c}$? None, in general.