MAU11602 Assignment 7, Due Wednesday 27 March 2024
Solutions

1. Prove the set identity $[[A \setminus [A \setminus B]] = [A \cap B]]$.
Hint: You can use the same method which was used in the notes for $[[[A \cup B] \cup C]] = [A \cup [B \cup C]]$:

   - Convert the equation to a pair of inclusions.
   - Write each inclusion as a substitution instance of a statement from zeroeth order logic.
   - Show that those statements are tautologies.

   In the example in the notes I skipped the last step, since I'd already given a number of examples of showing that statements are tautologies, but you shouldn't skip that step for this problem. A bit of preliminary work with the rules of inference for zeroeth order logic can simplify the statements you need to prove.
Solution:
$$[x \in [[A \setminus [A \setminus B]]$$
if and only if
$$[[x \in A] \wedge [\neg[x \in [A \setminus B]]]].$$
Similarly $[x \in [A \setminus B]$ if and only if $[[x \in A] \wedge [\neg[x \in B]]]$. So
$$[x \in [[A \setminus [A \setminus B]]$$
if and only if
$$[[x \in A] \wedge [\neg[[x \in A] \wedge [\neg[x \in B]]]]].$$
Using one of DeMorgan's laws, this is equivalent to
$$[[x \in A] \wedge [[\neg[x \in A]] \vee [\neg[\neg[x \in B]]]]]$$
and using the rule for double negation this is equivalent to
$$[[x \in A] \wedge [[\neg[x \in A]] \vee [x \in B]]].$$
Similarly, but much more easily,
$$[x \in [A \cap B]]$$
if and only if
$$[[x \in A] \wedge [x \in B]].$$
By Extensionality what we need to show is that $[x \in [[A \setminus [A \setminus B]]$ if and only if $[x \in [A \cap B]]$, i.e. that
$$[[x \in [[A \setminus [A \setminus B]] \supset [x \in [A \cap B]]]$$

$$[[x \in [A \cap B]] \supset [x \in [[A \setminus [A \setminus B]]].$$

By what we've shown above these are equivalent to

$$[[[x \in A] \wedge [[\neg[x \in A]] \vee [x \in B]]] \supset [[x \in A] \wedge [x \in B]]]$$

and

$$[[[x \in A] \wedge [x \in B]] \supset [[x \in A] \wedge [[\neg[x \in A]] \vee [x \in B]]]].$$

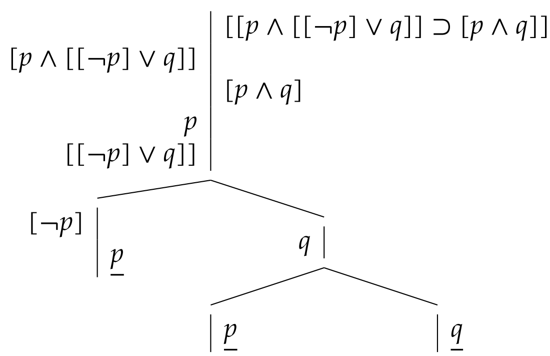
These are substitution instances of

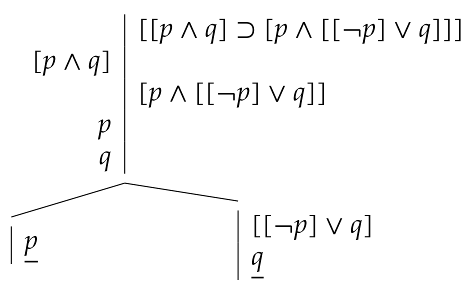
$$[[p \wedge [[\neg p] \vee q]] \supset [p \wedge q]]$$

and

$$[[p \wedge q] \supset [p \wedge [[\neg p] \vee q]]]$$

So it suffices to show that these are tautologies. You can do this with truth tables or with tableaux. I'll give the tableaux versions here.





2. In the notes I defined a minimal member of a set A to be a $C \in A$ such that if $B \in A$ and $B \subseteq C$ then $B = C$ and a maximal member of A to be $B \in A$ such that if $C \in A$ and $B \subseteq C$ then $B = C$. A related pair of notions are those of a least member and a greatest member. A least member of A is a $B \in A$ such that if $C \in A$ then $B \subseteq C$. A greatest member is a $C \in A$ such that if $B \in A$ then $B \subseteq C$.

   I further defined a set E to be finite if every non-empty set of subsets of E has a minimal and a maximal member.

(a) Give an example of a finite set E and a non-empty set A of subsets of E such that A has neither a least nor a greatest member.

(b) Show if a set A has a least member then that member is also minimal.

(c) Show if a set A has a greatest member then that member is also maximal.

(d) Show that if a set A has a least member then it has no other minimal member.

(e) Show that if a set A has a greatest member then it has no other maximal member.

(f) Suppose E is a finite set and A is a non-empty set of subsets of A. Show that if A has only one minimal member then that member is a least member.

Solution:

(a) Taking E to be $\{x, y\}$ where $x \neq y$ and A to be $\{\{x\}, \{y\}\}$ we have that $\{x\}$ and $\{y\}$ are neither least nor greatest members since we don't have either $\{x\} \subseteq \{y\}$ or $\{y\} \subseteq \{x\}$.

(b) Suppose C is a least member of A. If $B \in A$ and $B \subseteq C$ then $C \subseteq B$, because C is a least member, and so $B = C$, since $B \subseteq C$ and $C \subseteq B$. So $B \in A$ and $B \subseteq C$ imply $B = C$. In other words, C is minimal.

(c) Suppose B is a greatest member of A. If $C \in A$ and $B \subseteq C$ then $C \subseteq B$, because B is a greatest member, and so $B = C$, since $B \subseteq C$ and $C \subseteq B$. So $C \in A$ and $B \subseteq C$ imply $B = C$. In other words, B is maximal.

(d) Suppose B is a least member of A and C is a minimal member of A. Then $B \subseteq C$, because $C \in A$ and B is a least member of A. But then $B \in A$ and $B \subseteq C$, so $B = C$, since C is a minimal member. So A has no minimal member other than B.

(e) Suppose C is a greatest member of A and B is a maximal member of A. Then $B \subseteq C$, because $B \in A$ and C is a greatest member of A. But then $C \in A$ and $B \subseteq C$, so $B = C$, since B is a maximal member. So A has no maximal member other than C.

(f) Suppose C is the minimal member of A and H is some other member. Let B be the set of all $D \in A$ such that $D \subseteq H$. Then $H \in B$ so B is a non-empty set of subsets of E. E is finite so B must have a minimal member. Let G be a minimal member of B. If $F \in A$ and $F \subseteq G$ then $F \subseteq H$, since $G \in B$ and hence $G \subseteq H$. But then $F \in B$ and therefore $F = G$, since G is a minimal member of B. So if $F \in A$ and $F \subseteq G$ then $F = G$. From $G \in B$ it also follows that $G \in A$. So $G \in A$ and if $F \in A$ and $F \subseteq G$ then $F = G$. In other words G is a minimal member of A. But C is the only minimal member of A, so $G = C$. From $G \subseteq H$ it therefore follows that $C \subseteq H$. H was an arbitrary member of A so $C \subseteq H$ for all $H \in A$. In other words, C is minimal.