MAU11602 Assignment 6, Due Wednesday 20 March 2024
Solutions

1. (a) The set of natural numbers is, trivially, a subset of the set of natural numbers. Is it an arithmetic subset?

   (b) Is the union of any two arithmetic subsets of the natural numbers an arithmetic subset?

   (c) Is the intersection of any two arithmetic subsets of the natural numbers an arithmetic subset?

   (d) Is every subset of an arithmetic subset arithmetic?

   Note: You shouldn't need any set theory for this beyond the basic definitions. In any cases where your answer is "yes" give an informal proof. In any cases where your answer is "no" give a counterexample.
   Solution:

   (a) Yes. We just need to find an expression with one free variable which is satisfied by every natural number. There are many such expressions but the simplest is $(x = x)$.

   (b) Yes. If A is the set of natural numbers which when substituted for x into the expression P, where x is the only free variable in P, make the statement true and similarly B is the set of natural numbers which when substituted for x into the expression Q, where x is the only free variable in Q, make the statement true then x is the only free variable in $(P \vee Q)$ and $A \cup B$ is the set of natural numbers which when substituted for x into the expression $(P \vee Q)$ make that statement true.

   (c) Yes again. Just take the previous answer and replace each $\cup$ with a $\cap$ and each $\vee$ with a $\wedge$.

   (d) No. If this were true then since the set of natural numbers is itself arithmetic we could conclude that every subset of it is arithmetic. We know there are non-arithmetic sets though. Tarski's theorem, for example, says that the set of encodings of true statements is not arithmetic.